Matrix Theory — Exam 1 **MAT 335, Spring 2026 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) $BA+C$ b) AB c) $2CB+B$

$$A = \begin{bmatrix} 4 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & -3 \\ -2 & -4 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$$

2. (8pts) The solution of a linear system in four variables is given below in vector form.
a) Write a system of equations in usual form (variables left, constants right) that has this solution. (Don't do much - simply reverse the last step in the process of solving a system.)
b) Write the augmented matrix of the system.

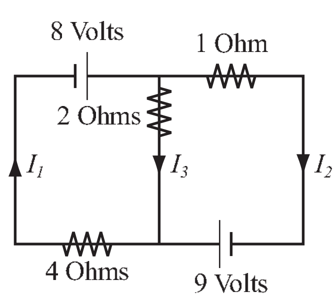
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \; x_3 \text{ free.}$$

3. (8pts) If A_θ is the 2×2 rotation matrix for a counterclockwise rotation around the origin by angle θ, show that $A_{\frac{\pi}{3}} A_{\frac{\pi}{2}} = A_{\frac{5\pi}{6}}$ using actual numbers in the matrices.

4. (14pts) A system of linear equations is given below.
a) Use the Gaussian elimination to solve the system.
b) Write the solution in vector form.

$$\begin{cases} x_1 & +3x_2 & +2x_3 & -2x_4 & = & 11 \\ -x_1 & -3x_2 & -x_3 & +2x_4 & = & -8 \\ 2x_1 & +6x_2 & +3x_3 & -4x_4 & = & 19 \end{cases}$$

5. (10pts) Determine the currents in each branch of the electrical circuit.



6. (12pts) Below is the augmented matrix of a system of linear equations. Determine the coefficient c for which the system has: a) no solutions, b) one solution, c) infinitely many solutions. (Note: no row operations are needed.)

$$A = \left[\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & c+4 & -2 & 0 \\ 0 & 0 & c^2+2c-8 & c-2 \end{array}\right]$$

7. (6pts) Find the elementary matrix E so that $EA = B$.

$$A = \begin{bmatrix} 0 & -2 & 7 \\ 3 & 2 & -1 \\ -4 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -2 & 7 \\ -5 & 14 & -1 \\ -4 & 6 & 0 \end{bmatrix} \qquad E =$$

8. (12pts) Consider the vectors $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 8 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$.

a) Do they span $\mathbf{R}^3$?
b) If not, which of the vectors can be removed so that the remaining two have the same span as the original three?

9. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\mathbf{a}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\mathbf{v}$ is in Span$\{\mathbf{u}, \mathbf{a}\}$.
b) A 2×3 matrix can have rank equal to 0, 1, 2, or 3.
c) For 2×2 matrices A and B, where A is diagonal, $AB = BA$.

Bonus. (10pts) Show: if $c \neq 0$, then Span$\{\mathbf{u}, \mathbf{v}\} = $ Span$\{\mathbf{u}, \mathbf{u} + c\mathbf{v}\}$.

Matrix Theory — Exam 2
MAT 335, Spring 2026 — D. Ivanšić

Name: ____________________________

Show all your work!

1. (10pts) Matrix A is given below. Evaluate its determinant by any (efficient) method.

$$\begin{vmatrix} 3 & 1 & -1 & 0 \\ 6 & 2 & 2 & -3 \\ -6 & 1 & 2 & -4 \\ 0 & -1 & -1 & 1 \end{vmatrix} =$$

2. (8pts) Determine values of c for which the matrix is not invertible.

$$A = \begin{bmatrix} c-1 & 1 & 3 \\ c-1 & 3 & 8 \\ 0 & 2 & c+7 \end{bmatrix}$$

3. (12pts) The matrix A is given below.
a) Find the inverse of A.
b) Use the inverse to easily solve the system below.

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\begin{aligned} 4x_1 & \; +3x_2 & = 0 \\ 3x_1 & \; +2x_2 & = -3 \end{aligned}$$

4. (14pts) Find the standard matrix of the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ and determine whether T is a) one-to-one, or b) onto.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 10x_2 + 13x_3 \\ 2x_1 + 7x_2 + 8x_3 \end{bmatrix}$$

5. (10pts) For a function $T : \mathbf{R}^2 \to \mathbf{R}^2$ the following is known:

a) T is a linear transformation

b) $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the null space of T.

Find the standard matrix of T.

6. (14pts) Let $V = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix} \right\}$

a) Show the vector $\begin{bmatrix} -7 \\ -1 \\ -9 \end{bmatrix}$ is in this subspace.

b) Find a basis for V that includes the vector from a).

7. (14pts) The set W is defined below.

a) Use the definition to show W is a subspace of $\mathbf{R}^3$.

b) Give a basis for W.

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid x_1 - 3x_2 + 4x_3 = 0 \right\}$$

8. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$ and T is a linear transformation, then $T(\mathbf{w}) \in \text{Span}\{T(\mathbf{u}), T(\mathbf{v})\}$
b) For a 3×3 matrix A, if $A^T = -A$, then $\det(A^T A) \leq 0$.
c) The set $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^2 \mid x_1 \cdot x_2 \geq 0 \right\}$ is a subspace of $\mathbf{R}^2$.

Bonus. (10pts) Let 0, A, B and C be 2×2 matrices, where 0 is a zero matrix. Let D be the 4×4 block matrix below, built using 0, A, B and C. Show $\det D = \det A \cdot \det B$. *(Hint: Only write out entries in one of the matrices A or B and expand by a row or a column.)*

$$D = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

Matrix Theory — Exam 3 **MAT 335, Spring 2026 — D. Ivanšić**

Name: ______________________________

Show all your work!

1. (18pts) For the matrix A, determine the dimensions of
a) Row A b) Col A c) Null A d) Null A^T.
Then give a basis for e) Col A f) Null A.

$$A = \begin{bmatrix} 2 & 1 & 7 & 0 \\ -3 & 2 & -7 & -7 \\ 5 & -2 & 13 & 9 \end{bmatrix}$$

2. (6pts) The vector is an eigenvector for the linear operator below. Determine the eigenvalue it corresponds to.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + 9x_2 + 8x_3 \\ -2x_1 - x_2 - 2x_3 \\ 2x_1 - 3x_2 - 2x_3 \end{bmatrix} \quad \text{vector: } \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

3. (20pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find the basis of the corresponding eigenspace.

$$A = \begin{bmatrix} 11 & -12 & 12 \\ 6 & -7 & 12 \\ 0 & 0 & 5 \end{bmatrix}$$

4. (8pts) Show that the matrix has no real eigenvalues.

$$D = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

5. (10pts) A 4×4 matrix A has eigenvalues -3, -2 and 1 and the dimension of the eigenspace corresponding to eigenvalue -3 is 2.
a) Determine the characteristic polynomial of A and justify.
b) Use the characteristic polynomial to evaluate $\det A$.

6. (20pts) The matrix A is given below.
a) Determine a basis for Row A and one for Null A.
b) Show that the union of the two bases you found in a) is a basis for $\mathbf{R}^3$.
c) For an $n \times n$ matrix A it is true the union of bases for Row A and Null A is a basis for $\mathbf{R}^n$. To see this is plausible, use dimensions of relevant subspaces to show that for a general $n \times n$ matrix, the union of the bases has n elements

$$A = \begin{bmatrix} 3 & 2 & 6 \\ 5 & 2 & 14 \\ 1 & 4 & -8 \end{bmatrix}$$

7. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If A is an $n \times n$ matrix, then nullity $A =$ nullity A^T.

b) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

c) If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for a matrix A, then $\mathbf{u} + \mathbf{v}$ is an eigenvector for A.

Bonus. (10pts) Find the eigenvalues of the matrix. Brute force will probably work poorly: use some row or column operations, as well as factoring out a common factor in a row or column.

$$A = \begin{bmatrix} -1 & 4 & -4 & -4 \\ 5 & -2 & 1 & 6 \\ 0 & 0 & -1 & 0 \\ 5 & -5 & 5 & 9 \end{bmatrix}$$

Matrix Theory — Final Exam
MAT 335, Spring 2026 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) BA b) CB^T c) ACA

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -7 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

2. (20pts) For the matrix A, determine the dimensions of
a) Row A b) Col A c) Null A d) Null A^T.
Then give a basis for e) Col A f) Row A g) Null A.

$$A = \begin{bmatrix} 1 & -2 & 0 & 2 \\ -1 & 3 & 1 & -3 \\ 2 & -2 & 3 & 1 \end{bmatrix}$$

3. (12pts) Below is the augmented matrix of a system of linear equations. Determine the coefficient b for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. (Note: no row operations are needed.)

$$A = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & b^2 - 3b & 0 & b \\ 0 & 0 & b+1 & 4 \end{array}\right]$$

4. (12pts) Let V be the subspace spanned by the vectors below, and $\mathbf{u}$ the additional vector.

a) Show that $\mathbf{u}$ is in V.

b) Determine a basis for V that consists of $\mathbf{u}$, the first vector in the set S, and possibly additional vectors from the set. (You do not need to do a lot more after a). If $\mathbf{u}$ can be written as a linear combination of some vectors from the set — and a) easily gives you the coefficients — then some vector from the set can be easily written as a linear combination of $\mathbf{u}$ and other vectors from the set.)

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ -3 \end{bmatrix} \right\} \qquad \mathbf{u} = \begin{bmatrix} 9 \\ -3 \\ 2 \\ -7 \end{bmatrix}$$

5. (12pts) Matrix A is given below.
a) Evaluate its determinant by any (efficient) method.
b) State if A is invertible and justify.

$$\begin{vmatrix} 0 & 4 & -1 & 1 \\ -3 & 1 & 1 & 2 \\ 1 & 0 & -2 & 3 \\ 2 & 3 & 0 & 1 \end{vmatrix} =$$

6. (6pts) Write the rotation matrix for a clockwise rotation around the origin by angle $\frac{\pi}{3}$ and use it find where the point $(4, 2)$ lands after it is rotated.

7. (14pts) The matrix A is given below.
a) Find the inverse of A.
b) Use the inverse to easily solve the system below.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 1 \\ -1 & 4 & 4 \end{bmatrix}$$

$$\begin{array}{rrrl} x_1 & +x_2 & & = 7 \\ 3x_1 & +4x_2 & +x_3 & = 3 \\ -x_1 & +4x_2 & +4x_3 & = -5 \end{array}$$

8. (10pts) Find the standard matrix of the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ and determine whether T is a) one-to-one, or b) onto.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 - 7x_3 \\ x_1 - 3x_2 + 3x_3 \end{bmatrix}$$

9. (12pts) The set W is defined below.
a) Use the definition to show W is a subspace of $\mathbf{R}^3$.
b) Give a basis for W.

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid 3x_1 + 2x_2 + x_3 = 0 \right\}$$

10. (24pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\mathbf{a}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\mathbf{v}$ is in Span$\{\mathbf{u}, \mathbf{a}\}$.
b) If $\mathbf{w} \in$ Span $\{\mathbf{u}, \mathbf{v}\}$ and T is a linear transformation, then $T(\mathbf{w}) \in$ Span $\{T(\mathbf{u}), T(\mathbf{v})\}$
c) For a 3×4 matrix A, its associated linear transformation T_A is not one-to-one.
d) If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for a matrix A, then $\mathbf{u} + \mathbf{v}$ is an eigenvector for A.

11. (16pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find the basis of the corresponding eigenspace.

$$A = \begin{bmatrix} 6 & 0 & 0 \\ -2 & 3 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

Bonus. (10pts) Find the eigenvalues of the matrix. Brute force will probably work poorly: use some row or column operations, as well as factoring out a common factor in a row or column.

$$A = \begin{bmatrix} -1 & 4 & -4 & -4 \\ 5 & -2 & 1 & 6 \\ 0 & 0 & -1 & 0 \\ 5 & -5 & 5 & 9 \end{bmatrix}$$