Matrix Theory — Final Exam
MAT 335, Fall 2022 — D. Ivanšić

Name: _____

Show all your work!

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) $CA + B$ b) $B^T C$ c) $A^T C C^T$

$$A = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -7 & -1 \\ 3 & 0 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 4 \\ -3 & 1 \end{bmatrix}$$

2. (18pts) A matrix A is given at right.
a) Find a basis for the nullspace of A.
b) Find a basis for the column space of A.
c) Find a basis for the row space of A.
d) Among Row A, Col A and Null A, are any two orthogonal, and if so, which ones?
(Note: only one Gauss elimination method is needed.)

$$A = \begin{bmatrix} 1 & -1 & -3 & 1 \\ -2 & 1 & 5 & 0 \\ 4 & -2 & -10 & 1 \end{bmatrix}$$

3. (12pts) Below is the augmented matrix of a system of linear equations. Determine the coefficient b for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. (Note: no row operations are needed.)

$$A = \left[\begin{array}{ccc|c} 1 & -2 & 7 & 1 \\ 0 & b & 0 & 4 \\ 0 & 0 & b^2+b & b \end{array}\right]$$

4. (10pts) Consider the vectors at right.
a) Are they linearly independent?
b) Are they orthogonal?
c) Are they a basis for $\mathbf{R}^n$?

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

5. (12pts) Matrix A is given below.
a) Evaluate its determinant by any (efficient) method.
b) State if A is invertible and justify.

$$\begin{vmatrix} 0 & 3 & 0 & 1 \\ -1 & 4 & 2 & -3 \\ 2 & -2 & 3 & -2 \\ -3 & 9 & 2 & 1 \end{vmatrix} =$$

6. (6pts) Write the rotation matrix for a counterclockwise rotation around the origin by angle $\frac{3\pi}{4}$ and use it find where the point $(-1, 3)$ lands after it is rotated.

7. (12pts) The matrix A is given below.
a) Find the inverse of A.
b) Use the inverse to easily solve the system below.

$$A = \begin{bmatrix} -2 & 3 \\ -1 & 5 \end{bmatrix}$$

$$\begin{aligned} -2x_1 \quad +3x_2 \quad &= 3 \\ -x_1 \quad +5x_2 \quad &= -4 \end{aligned}$$

8. (10pts) Find the standard matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ and determine whether T is a) one-to-one, or b) onto.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + 4x_2 \\ x_1 - 3x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$$

9. (18pts) The set W is defined below.
a) Use the definition to show W is a subspace of $\mathbf{R}^3$.
b) Give a set of generating vectors for W.
c) Find a basis for $W^{\perp}$.

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid x_1 - x_2 + 2x_3 = 0 \right\}$$

10. (24pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent and T is a *nonzero* linear transformation, then $\{T(\mathbf{u}_1), T(\mathbf{u}_2)\}$ is linearly independent.
b) If A is a 4×3 matrix with rank 2, then the columns of A are linearly dependent.
c) If the characteristic polynomial of a 2×2 matrix A is $(t-3)^2$, then $A = 3I$.
Justify your answer by giving a logical argument or a counterexample.
d) If -2 and 3 are eigenvalues of a 2×2 matrix A, then A is invertible.

11. (16pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find the basis of the corresponding eigenspace.

$$A = \begin{bmatrix} -3 & 3 & 9 \\ 0 & 3 & 0 \\ -2 & 1 & 6 \end{bmatrix}$$

Bonus. (10pts) Show: if vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly independent.