Matrix Theory — Exam 3
MAT 335, Fall 2022 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (12pts) Give a basis for the row space of A and state the dimension of Null A.

$$A = \begin{bmatrix} 2 & 5 & -3 & 0 \\ 1 & 3 & 5 & -4 \\ 7 & 19 & 9 & -12 \end{bmatrix}$$

2. (8pts) One of the vectors is an eigenvector for the matrix A below. Determine which one, and the eigenvalue it corresponds to.

$$A = \begin{bmatrix} 2 & -6 & 6 \\ 1 & 9 & -6 \\ -2 & 16 & -13 \end{bmatrix} \quad \text{vectors: } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

3. (14pts) Let W be the subspace of $\mathbf{R}^4$ spanned by the set given below. Find a basis for $W^\perp$.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

4. (16pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find the basis of the corresponding eigenspace.

$$A = \begin{bmatrix} -6 & -8 & 0 \\ 3 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

5. (12pts) a) Verify that the vectors at left are an orthonormal set.
b) The vector at right is in the subspace spanned by the vectors. Write it as a linear combination of the vectors from the orthonormal set. (Avoid solving a system: use the fact the set is orthonormal.)

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\} \qquad \begin{bmatrix} 7 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

6. (10pts) A 3×3 matrix A has eigenvalues 1, and -3, and the dimension of the eigenspace corresponding to eigenvalue 1 is 2.
a) Determine the characteristic polynomial of A and justify.
b) Use the characteristic polynomial to evaluate $\det(A - 5I)$.

7. (10pts) Show: the vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal if and only if $|\mathbf{u}| = |\mathbf{v}|$. (Do not use coordinates. I beseech you.)

8. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If 0 is an eigenvalue of A, then A is not invertible.

b) If the characteristic polynomial of a 2×2 matrix A is $(t-3)^2$, then $A = 3I$.

c) If A is 2×2 matrix, then $(A\mathbf{x})\cdot(A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{R}^2$.

Bonus. (10pts) An $n \times n$ matrix A is called orthogonal if $A^T A = I_n$. Prove the following statements.

a) A 2×2 rotation matrix is orthogonal.

b) The columns of A are an orthonormal basis for $\mathbf{R}^n$.

c) Multiplying by A preserves the norm of a vector, that is, $|A\mathbf{x}| = |\mathbf{x}|$.

(Hint: show $|A\mathbf{x}|^2 = |\mathbf{x}|^2$*.)*