Matrix Theory — Final Exam MAT 335, Spring 2026 — D. Ivanšić

Name: Saul Ocean

Show all your work!

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) BA b) CB^T c) ACA

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -7 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

a) $\begin{bmatrix} 2 & -7 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 6-7 & 4+21 \\ 9+0 & 6+0 \\ -3+4 & -2-12 \end{bmatrix} = \begin{bmatrix} -1 & 25 \\ 9 & 6 \\ 1 & -14 \end{bmatrix}$

b) $\begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ -7 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -4-7 & -6+0 & 2+4 \end{bmatrix} = \begin{bmatrix} -11 & -6 & 6 \end{bmatrix}$

c) $\overbrace{\begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 & 1 \end{bmatrix}}^{\text{not defined}} \begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix}$

2×2 1×2

2. (20pts) For the matrix A, determine the dimensions of
a) Row A b) Col A c) Null A d) Null A^T.
Then give a basis for e) Col A f) Row A g) Null A.

$$A = \begin{bmatrix} 1 & -2 & 0 & 2 \\ -1 & 3 & 1 & -3 \\ 2 & -2 & 3 & 1 \end{bmatrix} \begin{matrix} \\ \cdot 1 \\ \cdot(-2) \end{matrix} \sim \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \cdot(-1)$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \cdot 2 \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$ Rank $=3$

a) dim Row A = rank $A = 3$ b) dim Col A = rank $A = 3$ c) dim Null $A = 4 - \text{rank} A = 1$

d) dim null $A^T = 3 - \text{rank} A^T = 3 - \text{rank} A = 0$

$x_1 = -2x_4$
$x_2 = 0$
$x_3 = x_4$

e) Col A basis

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$

f) Row A basis

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

g) Null A basis

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_4 \\ 0 \\ x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

basis for Null A

3. (12pts) Below is the augmented matrix of a system of linear equations. Determine the coefficient b for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. (Note: no row operations are needed.)

$$A = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & b^2-3b & 0 & b \\ 0 & 0 & b+1 & 4 \end{array}\right]$$

a) one solution if $b^2-3b\neq 0$ and $b\neq 0$ and $b+1\neq 0$ $\quad b(b-3)=0 \quad b=0,3$ and $b\neq 0$
$b+1=0 \quad b=-1$

One solution if $b\neq -1, 3$

b) Infinitely many solutions $b^2-3b=0$ and $b=0$
$b=0,3$ and $b=0$

if $b=0$

c) No solutions: if $b+1=0$ or ($b^2-3b=0$ and $b\neq 0$)
$b=-1 \qquad b=0,3$ and $b\neq 0$

No solutions if $b=-1$ or $b=3$

4. (12pts) Let V be the subspace spanned by the vectors below, and $\mathbf{u}$ the additional vector.

a) Show that $\mathbf{u}$ is in V.

b) Determine a basis for V that consists of $\mathbf{u}$, the first vector in the set S, and possibly additional vectors from the set. (You do not need to do a lot more after a). If $\mathbf{u}$ can be written as a linear combination of some vectors from the set — and a) easily gives you the coefficients — then some vector from the set can be easily written as a linear combination of $\mathbf{u}$ and other vectors from the set.)

$$S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec u_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec u_2}, \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec u_3}, \underbrace{\begin{bmatrix} 3 \\ -1 \\ 0 \\ -3 \end{bmatrix}}_{\vec u_4} \right\} \qquad \mathbf{u} = \begin{bmatrix} 9 \\ -3 \\ 2 \\ -7 \end{bmatrix}$$

a) $$\begin{bmatrix} 1 & -1 & 1 & 3 & 9 \\ 0 & 1 & -1 & -1 & -3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 3 & 9 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 1 & 0 & -3 & -7 \\ 0 & 1 & 0 & -3 & -7 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 3 & 9 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 1 & -2 & -4 \end{array}\right]$$

Dim $V=3$, First three vectors are a basis and $\vec u\in V$ because the system $A\vec x=\vec u$ has a solution

b) $$\sim \begin{bmatrix} 1 & -1 & 0 & 5 & 13 \\ 0 & 1 & 0 & -3 & -7 \\ 0 & 0 & 1 & -2 & -4 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & -3 & -7 \\ 0 & 0 & 1 & -2 & -4 \end{array}\right]$$

It follows that
$\vec u = 6\vec u_1 - 7\vec u_2 - 4\vec u_3$
So $\vec u_3 = \frac{1}{4}(-\vec u + 6\vec u_1 - 7\vec u_2)$

Thus, $V = \text{span}\{\vec u_1, \vec u_2, \vec u_3\} \overset{\text{since } \vec u\in V}{=} \text{span}\{\vec u_1, \vec u_2, \vec u_3, \vec u\} \overset{\text{since } \vec u_3 \text{ is a lin. comb of } \vec u_1, \vec u_2, \vec u}{=} \text{span}\{\vec u_1, \vec u_2, \vec u\}$

Since $\dim V = 3$ it follows that $\{\vec u_1, \vec u_2, \vec u\}$ are lin. indep., hence a basis for V

5. (12pts) Matrix A is given below.
a) Evaluate its determinant by any (efficient) method.
b) State if A is invertible and justify.

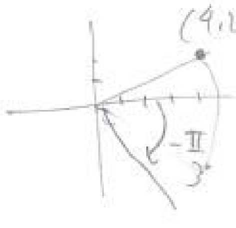
$$\begin{vmatrix} 0 & 4 & -1 & 1 \\ -3 & 1 & 1 & 2 \\ 1 & 0 & -2 & 3 \\ 2 & 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 4 & -1 & 1 \\ 0 & 1 & -5 & 11 \\ 1 & 0 & -2 & 3 \\ 0 & 3 & 4 & -5 \end{vmatrix} = (-1)^{3+1}\cdot 1 \begin{vmatrix} 4 & -1 & 1 \\ 1 & -5 & 11 \\ 3 & 4 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 19 & -43 \\ 1 & -5 & 11 \\ 0 & 19 & -38 \end{vmatrix} = (-1)^{2+1}\left(19\cdot(-38) - 19(-43)\right) = -19(-38+43)$$

$$= -19\cdot 5 = -95$$

b) Since $\det A \neq 0$, the matrix is invertible

6. (6pts) Write the rotation matrix for a clockwise rotation around the origin by angle $\frac{\pi}{3}$ and use it find where the point $(4,2)$ lands after it is rotated.



$$R = \begin{bmatrix} \cos(-\frac{\pi}{3}) & -\sin(-\frac{\pi}{3}) \\ \sin(-\frac{\pi}{3}) & \cos(-\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} + \frac{2\sqrt{3}}{2} \\ -\frac{4\sqrt{3}}{2} + \frac{2}{2} \end{bmatrix} = \begin{bmatrix} \frac{4+2\sqrt{3}}{2} \\ \frac{-4\sqrt{3}+2}{2} \end{bmatrix} = \begin{bmatrix} 2+\sqrt{3} \\ 1-2\sqrt{3} \end{bmatrix}$$

7. (14pts) The matrix A is given below.
a) Find the inverse of A.
b) Use the inverse to easily solve the system below.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 1 \\ -1 & 4 & 4 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad +x_2 \quad\quad\quad &= 7 \\ 3x_1 \quad +4x_2 \quad +x_3 &= 3 \\ -x_1 \quad +4x_2 \quad +4x_3 &= -5 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 & 1 & 0 \\ -1 & 4 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 5 & 4 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 0 & -1 & 16 & -5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -16 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -13 & -4 & 1 \\ 0 & 0 & 1 & -16 & 5 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -12 & 4 & -1 \\ 0 & 1 & 0 & 13 & -4 & 1 \\ 0 & 0 & 1 & -16 & 5 & -1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -12 & 4 & -1 \\ 13 & -4 & 1 \\ -16 & 5 & -1 \end{bmatrix}$$

Sol. to $A\vec{x} = \vec{b}$ is $A^{-1}\vec{b}$ if A is invertible

$$\vec{x} = \begin{bmatrix} -12 & 4 & -1 \\ 13 & -4 & 1 \\ -16 & 5 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -84+12+5 \\ 91-12-5 \\ -112+15+5 \end{bmatrix} = \begin{bmatrix} -67 \\ 74 \\ -92 \end{bmatrix}$$

8. (10pts) Find the standard matrix of the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ and determine whether T is a) one-to-one, or b) onto.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 - 7x_3 \\ x_1 - 3x_2 + 3x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & -7 \\ 1 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 2 & 3 & -7 \end{bmatrix} \begin{matrix} 2 \\ \cdot(-1) \end{matrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 9 & -13 \end{bmatrix} \leftarrow \text{rank} = 2$$

a) nullity $A = 3 - \text{rank } A = 1$ since nullity $A \neq 0$, T is not one-to-one

b) since rank $A = \dim \mathbb{R}^2$, T is onto

9. (12pts) The set W is defined below.
a) Use the definition to show W is a subspace of $\mathbf{R}^3$.
b) Give a basis for W.

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid 3x_1 + 2x_2 + x_3 = 0 \right\}$$

Let $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in W$

Then $3u_1 + 2u_2 + u_3 = 0$
$3v_1 + 2v_2 + v_3 = 0$

$3u_1 + 3v_1 + 2u_2 + 2v_2 + u_3 + v_3 = 0$
$3(u_1 + v_1) + 2(u_2 + v_2) + (u_3 + v_3) = 0$

so $\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$ satisfies the equation so it is in W

$c \cdot 3u_1 + c \cdot 2u_2 + c \cdot u_3 = 0$
$3(cu_1) + 2(cu_2) + cu_3 = 0$

$\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ satisfies the equation so is in W

W is a subspace.

b) $3x_1 = -2x_2 - x_3$
$x_1 = -\frac{2}{3}x_2 - \frac{1}{3}x_3$
x_2, x_3 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_2 - \frac{1}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

basis for W.

10. (24pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If $\mathbf{a}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\mathbf{v}$ is in Span$\{\mathbf{u}, \mathbf{a}\}$.

b) If $\mathbf{w} \in$ Span $\{\mathbf{u}, \mathbf{v}\}$ and T is a linear transformation, then $T(\mathbf{w}) \in$ Span $\{T(\mathbf{u}), T(\mathbf{v})\}$

c) For a 3×4 matrix A, its associated linear transformation T_A is not one-to-one.

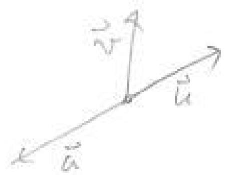
d) If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for a matrix A, then $\mathbf{u} + \mathbf{v}$ is an eigenvector for A.

a) False

$\vec{a} = -2\vec{u}$ so $\vec{a} \in \text{Span}\{\vec{u}, \vec{v}\}$ but $\vec{v} \notin \text{Span}\{\vec{u}, \vec{a}\}$

$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not in Span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right\}$



b) True If $\vec{w} \in \text{Span}\{\vec{u}, \vec{v}\}$ $\vec{w} = a\vec{u} + b\vec{v}$ so $T(\vec{w}) = T(a\vec{u} + b\vec{v})$
$= T(a\vec{u}) + T(b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$ so $T(\vec{w}) \in \text{Span}\{T(\vec{u}), T(\vec{v})\}$

c) True: nullity $A = 4 - \text{rank } A$. Since rank $A \leq 3$, $4 - \text{rank} A \geq 1 > 0$
Hence, nullity $A \neq 0$, so T_A is not one-to-one

d) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Then $\vec{e}_1$ is an eigenvector for $\lambda = 1$
$\vec{e}_2$ is an eigenvector for $\lambda = 0$

$A(\vec{e}_1 + \vec{e}_2) = \vec{e}_1 + \vec{0} = \vec{e}_1$, not a multiple of $\vec{e}_1 + \vec{e}_2$.

11. (16pts) The matrix A is given below.

a) Find the eigenvalues for the matrix.

b) For each eigenvalue, find the basis of the corresponding eigenspace.

$$A = \begin{bmatrix} 6 & 0 & 0 \\ -2 & 3 & 5 \\ 1 & 3 & 1 \end{bmatrix} \qquad \det(A-\lambda I) = \begin{vmatrix} 6-\lambda & 0 & 0 \\ -2 & 3-\lambda & 5 \\ 1 & 3 & 1-\lambda \end{vmatrix} = (6-\lambda)\begin{vmatrix} 3-\lambda & 5 \\ 3 & 1-\lambda \end{vmatrix}$$

$$= (6-\lambda)\big(\underbrace{(3-\lambda)(1-\lambda)}_{(\lambda-3)(\lambda-1)} - 15\big) = (6-\lambda)(\lambda^2-4\lambda+3-15)$$

$$= -(\lambda-6)(\lambda^2-4\lambda-12)$$

$$= -(\lambda-6)(\lambda-6)(\lambda+2) = -(\lambda-6)^2(\lambda+2)$$

We see the eigenvalues are $\lambda = 6, -2$

Eigenspace of $\lambda=6$

$$A-6I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & 5 \\ 1 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{3} \end{bmatrix}$$

$x_1 = 0$

$x_2 = \frac{5}{3}x_3$ $\quad x_3$ free so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix} x_3$ $\quad$ Basis $= \left\{ \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix} \right\}$

Eigenspace for $\lambda = -2$

$$A+2I = \begin{bmatrix} 8 & 0 & 0 \\ -2 & 5 & 5 \\ 1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 5 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$x_1 = 0$

$x_2 = -x_3$ $\quad$ so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} x_3$ $\quad$ Basis: $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Bonus. (10pts) Find the eigenvalues of the matrix. Brute force will probably work poorly: use some row or column operations, as well as factoring out a common factor in a row or column.

$$A = \begin{bmatrix} -1 & 4 & -4 & -4 \\ 5 & -2 & 1 & 6 \\ 0 & 0 & -1 & 0 \\ 5 & -5 & 5 & 9 \end{bmatrix} \qquad \begin{vmatrix} -1-\lambda & 4 & -4 & -4 \\ 5 & -2-\lambda & 1 & 6 \\ 0 & 0 & -1-\lambda & 0 \\ 5 & -5 & 5 & 9-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} -1-\lambda & 4 & -4 \\ 5 & -2-\lambda & 6 \\ 5 & -5 & 9-\lambda \end{vmatrix}$$

Eigenvalues: $\lambda = -1, 3, 4$

$$= (-1-\lambda)\begin{vmatrix} -1-\lambda & 3-\lambda & -4 \\ 5 & 3-\lambda & 6 \\ 5 & 0 & 9-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda)\begin{vmatrix} -1-\lambda & 1 & -4 \\ 5 & 1 & 6 \\ 5 & 0 & 9-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3)\begin{vmatrix} -6-\lambda & 0 & -10 \\ 5 & 1 & 6 \\ 5 & 0 & 9-\lambda \end{vmatrix}$$

$$= (\lambda+1)(\lambda-3)\begin{vmatrix} -6-\lambda & -10 \\ 5 & 9-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3)\big(\underbrace{(-6-\lambda)(9-\lambda)}_{(\lambda+6)(\lambda-9)} + 50\big) = (\lambda+1)(\lambda-3)(\lambda^2-3\lambda-54+50)$$

$$= (\lambda+1)(\lambda-3)(\lambda^2-3\lambda-4) = (\lambda+1)(\lambda-3)(\lambda-4)(\lambda+1)$$

$$= (\lambda+1)^2(\lambda-3)(\lambda-4)$$