Matrix Theory — Exam 3
MAT 335, Spring 2026 — D. Ivanšić

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Show all your work!

1. (18pts) For the matrix A, determine the dimensions of
a) Row A b) Col A c) Null A d) Null A^T.
Then give a basis for e) Col A f) Null A.

$$A = \begin{bmatrix} 2 & 1 & 7 & 0 \\ -3 & 2 & -7 & -7 \\ 5 & -2 & 13 & 9 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 0 & -7 \\ -3 & 2 & -7 & -7 \\ 5 & -2 & 13 & 9 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 & 7 \\ 0 & -7 & -7 & 14 \\ 0 & 13 & 13 & -26 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & 7 \\ 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

rank $A = 2$ So a),b) $\dim \text{Row} A = \dim \text{Col} A = 2$

c) $\dim \text{Null} A = 4 - \text{rank} A = 2$

d) $\dim \text{Null} A^T = 3 - \text{rank} A^T = 3 - \text{rank} A = 1$

e) Basis for Col A is pivot columns $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

f) Basis for Null A:

$x_1 + 3x_3 + x_4 = 0 \quad x_1 = -3x_3 - x_4$

$x_2 + x_3 - 2x_4 = 0 \quad x_2 = -x_3 + 2x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

2. (6pts) The vector is an eigenvector for the linear operator below. Determine the eigenvalue it corresponds to.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + 9x_2 + 8x_3 \\ -2x_1 - x_2 - 2x_3 \\ 2x_1 - 3x_2 - 2x_3 \end{bmatrix} \quad \text{vector: } \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 8 - 16 \\ -4 - (-4) \\ 4 - (-4) \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 8 \end{bmatrix} = -4 \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

eigenvalue is -4.

3. (20pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find the basis of the corresponding eigenspace.

$$A = \begin{bmatrix} 11 & -12 & 12 \\ 6 & -7 & 12 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{vmatrix} 11-\lambda & -12 & 12 \\ 6 & -7-\lambda & 12 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)\big((11-\lambda)(-7-\lambda)+72\big)$$

$$= -(\lambda-5)\big((\lambda-11)(\lambda+7)+72\big)$$

$$= -(\lambda-5)(\lambda^2-4\lambda-77+72)$$

$$= -(\lambda-5)(\lambda^2-4\lambda-5) = -(\lambda-5)(\lambda-5)(\lambda+1)$$

$$= -(\lambda-5)^2(\lambda+1)$$

Eigenvalues are $5, -1$

Eigenspace for 5:

$$\begin{bmatrix} 11-5 & -12 & 12 \\ 6 & -7-5 & 12 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 6 & -12 & 12 \\ 6 & -12 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$$

$x_1 - 2x_2 + 2x_3 = 0$

$x_1 = 2x_2 - 2x_3$, x_2, x_3 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

basis

Eigenspace for -1

$$\begin{bmatrix} 11+1 & -12 & 12 \\ 6 & -7+1 & 12 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 12 & -12 & 12 \\ 6 & -6 & 12 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_1 - x_2 = 0$ $x_1 = x_2$

$x_3 = 0$ x_2 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

4. (8pts) Show that the matrix has no real eigenvalues.

$$D = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & -3 \\ 1 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda)+3 = (\lambda-2)(\lambda-4)+3$$

$$= \lambda^2 - 6\lambda + 11$$

$\lambda^2 - 6\lambda + 11 = 0$

$$\lambda = \frac{-(-6) \pm \sqrt{(-6)^2 - 4\cdot 1\cdot 11}}{2\cdot 1} = \frac{6 \pm \sqrt{-8}}{2}$$

no real solutions

5. (10pts) A 4×4 matrix A has eigenvalues -3, -2 and 1 and the dimension of the eigenspace corresponding to eigenvalue -3 is 2.
a) Determine the characteristic polynomial of A and justify.
b) Use the characteristic polynomial to evaluate det A.

a) Must have factors $(\lambda+3), (\lambda+2), (\lambda-1)$ and multiplicity of $(\lambda+3)$ is ≥ 2
Since it is a polynomial of degree 4, it has to be 2
So $\det(A-\lambda I) = (\lambda+3)^2(\lambda+2)(\lambda-1) = P(\lambda)$

b) $\det A = P(0) = (0+3)^2(0+2)(0-1) = -18$

6. (20pts) The matrix A is given below.
a) Determine a basis for Row A and one for Null A.
b) Show that the union of the two bases you found in a) is a basis for $\mathbf{R}^3$.
c) For an $n \times n$ matrix A it is true the union of bases for Row A and Null A is a basis for $\mathbf{R}^n$. To see this is plausible, use dimensions of relevant subspaces to show that for a general $n \times n$ matrix, the union of the bases has n elements

$$A = \begin{bmatrix} 3 & 2 & 6 \\ 5 & 2 & 14 \\ 1 & 4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -8 \\ 3 & 2 & 6 \\ 5 & 2 & 14 \end{bmatrix} \begin{matrix} \\ \cdot(-3) \\ \cdot(-5) \end{matrix} \sim \begin{bmatrix} 1 & 4 & -8 \\ 0 & -10 & 30 \\ 0 & -18 & 54 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -8 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -8 \\ 0 & 1 & -3 \end{bmatrix} \cdot(-4)$$

$$\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$$

$x_1 + 4x_3 = 0 \quad x_1 = -4x_3$
$x_2 - 3x_3 = 0 \quad x_2 = 3x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

↑ Basis for Null A

Basis for Row A: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$

b) Show columns span $\mathbb{R}^3$. Since there are 3, it will be a basis

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 4 & -3 & 1 \end{bmatrix} \cdot(-4) \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & -3 & 17 \end{bmatrix} \cdot 3 \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 26 \end{bmatrix} \quad \text{rank} = 3$$

c) dim Row A = rank A
dim Null A = n − rank A } adding gives n

7. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If A is an $n \times n$ matrix, then nullity A = nullity A^T.

b) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

c) If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for a matrix A, then $\mathbf{u}+\mathbf{v}$ is an eigenvector for A.

a) True nullity $A = n - \text{rank} A = n - \text{rank} A^T = \text{nullity} A^T$

↑ columns of A   ↑ there are n columns of A^T

b) True. If λ is an eigenvalue, $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$

Then $A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$

c) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $A\vec{e}_1 = \vec{e}_1$, $A\vec{e}_2 = \vec{0}$

$\vec{e}_1$ eigenvector for 1

$\vec{e}_2$ eigenvector for 0

$A(\vec{e}_1 + \vec{e}_2) = A\vec{e}_1 + A\vec{e}_2 = \vec{e}_1 + \vec{0} = \vec{e}_1$

$\vec{e}_1$ is not parallel to $\vec{e}_1 + \vec{e}_2$, so $\vec{e}_1 + \vec{e}_2$ is not an eigenvector

Bonus. (10pts) Find the eigenvalues of the matrix. Brute force will probably work poorly: use some row or column operations, as well as factoring out a common factor in a row or column.

$$A = \begin{bmatrix} -1 & 4 & -4 & -4 \\ 5 & -2 & 1 & 6 \\ 0 & 0 & -1 & 0 \\ 5 & -5 & 5 & 9 \end{bmatrix}$$

$$\begin{vmatrix} -1-\lambda & 4 & -4 & -4 \\ 5 & -2-\lambda & 1 & 6 \\ 0 & 0 & -1-\lambda & 0 \\ 5 & -5 & 5 & 9-\lambda \end{vmatrix} \underset{\text{3rd row}}{\overset{\text{expand along}}{=}} (-1-\lambda)\begin{vmatrix} -1-\lambda & 4 & -4 \\ 5 & -2-\lambda & 6 \\ 5 & -5 & 9-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\begin{vmatrix} -1-\lambda & 3-\lambda & -4 \\ 5 & 3-\lambda & 6 \\ 5 & 0 & 9-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda)\begin{vmatrix} -1-\lambda & 1 & -4 \\ 5 & 1 & 6 \\ 5 & 0 & 9-\lambda \end{vmatrix} \overset{\text{(-1)}}{=} (-1-\lambda)(3-\lambda)\begin{vmatrix} -6-\lambda & 0 & -10 \\ 5 & 1 & 6 \\ 5 & 0 & 9-\lambda \end{vmatrix}$$

↑ 2nd column

$$= (-1-\lambda)(3-\lambda)\left((-6-\lambda)(9-\lambda)+50\right) = (-1-\lambda)(3-\lambda)(\lambda^2-3\lambda-4)$$

$(\lambda+6)(\lambda-9)$

$$= (\lambda+1)(\lambda-3)(\lambda-4)(\lambda+1) = (\lambda+1)^2(\lambda-3)(\lambda-4)$$

$\lambda = -1, 3, 4$