

1. (10pts) Matrix  $A$  is given below. Evaluate its determinant by any (efficient) method.

$$\begin{aligned}
 \begin{vmatrix} 3 & 1 & -1 & 0 \\ 6 & 2 & 2 & -3 \\ -6 & 1 & 2 & -4 \\ 0 & -1 & -1 & 1 \end{vmatrix} &= \begin{vmatrix} 3 & 1 & -1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 3 & 0 & -4 \\ 0 & -1 & -1 & 1 \end{vmatrix} = 3 \begin{vmatrix} 0 & 4 & -3 \\ 3 & 0 & -4 \\ -1 & -1 & 1 \end{vmatrix} \quad \begin{array}{l} \text{expand} \\ \text{along} \\ \text{1st row} \end{array} \\
 &= 3(-4(3-4) + (-3)(-3)) = 3(4+9) = 39
 \end{aligned}$$

2. (8pts) Determine values of  $c$  for which the matrix is not invertible.

$$\begin{aligned}
 A &= \begin{bmatrix} c-1 & 1 & 3 \\ c-1 & 3 & 8 \\ 0 & 2 & c+7 \end{bmatrix} \quad \begin{array}{l} \text{expand} \\ \text{along} \\ \text{column 1} \end{array} \quad \det A = (c-1)(3(c+7)-16) - (c-1)(c+7-6) \\
 &= (c-1)(3c+21-16 - (c+1)) \\
 &= (c-1)(2c+4) = 2(c-1)(c+2) \\
 \text{Not invertible} & \text{ when } \det A = 0 \\
 \text{that is, when} & \quad c = 1, -2 \\
 2(c-1)(c+2) &= 0
 \end{aligned}$$

3. (12pts) The matrix  $A$  is given below.

a) Find the inverse of  $A$ .

b) Use the inverse to easily solve the system below.

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\begin{aligned}
 4x_1 + 3x_2 &= 0 \\
 3x_1 + 2x_2 &= -3
 \end{aligned}$$

$$\begin{aligned}
 \left[ \begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array} \right] &\xrightarrow{2 \cdot (-1)} \sim \left[ \begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 3 & 2 & 0 & 1 \end{array} \right] \xrightarrow{2 \cdot (-3)} \sim \left[ \begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & -1 & -3 & 4 \end{array} \right] \xrightarrow{(-1)} \\
 &\sim \left[ \begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & -4 \end{array} \right] \xrightarrow{(-1)} \sim \left[ \begin{array}{cc|cc} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & -4 \end{array} \right] \\
 A^{-1} &= \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}
 \end{aligned}$$

$$\text{solution is } A^{-1}b = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -9 \\ 12 \end{bmatrix}$$

4. (14pts) Find the standard matrix of the linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  and determine whether  $T$  is a) one-to-one, or b) onto.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 10x_2 + 13x_3 \\ 2x_1 + 7x_2 + 8x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 10 & 13 \\ 2 & 7 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 7 & 8 \\ 3 & 10 & 13 \end{bmatrix} \xrightarrow{2 \cdot (-1)} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & -2 \end{bmatrix}$$

We see that  $\text{rank } A = 2$ , so  $T$  is onto

$\text{rank } A \neq 3$  (no of columns), so  $A$  is not one-to-one

5. (10pts) For a function  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the following is known:

a)  $T$  is a linear transformation

b)  $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is in the null space of  $T$ .

Find the standard matrix of  $T$ .

Need  $T(\vec{e}_1), T(\vec{e}_2)$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{e}_1 = \frac{1}{2}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$\vec{e}_2 = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$T(\vec{e}_1) = \frac{1}{2}\left(T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right) = \frac{1}{2}\left(\begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_2) = \frac{1}{2}\left(T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right) = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Matrix of  $T$  is

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

6. (14pts) Let  $V = \text{Span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix} \right\}$

a) Show the vector  $\begin{bmatrix} -7 \\ -1 \\ -9 \end{bmatrix}$  is in this subspace.

b) Find a basis for  $V$  that includes the vector from a).

a)  $\left[ \begin{array}{ccc|c} 3 & 2 & 8 & -7 \\ 1 & 2 & 4 & -1 \\ 2 & -3 & 1 & -9 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[ \begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 3 & 2 & 8 & -7 \\ 2 & -3 & 1 & -9 \end{array} \right] \xrightarrow{\substack{R_2 - 3R_1 \\ R_3 - 2R_1}} \left[ \begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & -4 & -4 & -4 \\ 0 & -7 & -7 & -7 \end{array} \right] \xrightarrow{\substack{R_2 \cdot (-\frac{1}{4}) \\ R_3 \cdot (-\frac{1}{7})}} \left[ \begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right] \xrightarrow{R_3 - R_2} \left[ \begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$

Rank = 2 so system has a solution and  $\begin{bmatrix} -7 \\ -1 \\ -9 \end{bmatrix}$  is in the subspace

b) The pivot columns are a basis for the span, hence, it has dimension 2. Any vector nonparallel to  $\begin{bmatrix} -7 \\ -1 \\ -9 \end{bmatrix}$  will make a lin. indep. set with  $\begin{bmatrix} -7 \\ -1 \\ -9 \end{bmatrix}$  so will be a basis for a 2-dimensional subspace. Hence,  $\left\{ \begin{bmatrix} -7 \\ -1 \\ -9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$  is a basis.

7. (14pts) The set  $W$  is defined below.

a) Use the definition to show  $W$  is a subspace of  $\mathbb{R}^3$ .

b) Give a basis for  $W$ .

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 - 3x_2 + 4x_3 = 0 \right\}$$

b)  $x_1 - 3x_2 + 4x_3 = 0$   
 $x_1 = 3x_2 - 4x_3$   $x_2, x_3$  free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Basis is  $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$

a) Suppose  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  and  $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$  are in  $W$

Then  $x_1 - 3x_2 + 4x_3 = 0$

add  $y_1 - 3y_2 + 4y_3 = 0$

$$x_1 + y_1 - 3x_2 - 3y_2 + 4x_3 + 4y_3 = 0$$

$$(x_1 + y_1) - 3(x_2 + y_2) + 4(x_3 + y_3) = 0$$

so  $\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$  is in  $W$

$$c x_1 - c 3 x_2 + c \cdot 4 x_3 = 0$$

$$c x_1 - 3(c x_2) + 4(c x_3) = 0$$

so  $\begin{bmatrix} c x_1 \\ c x_2 \\ c x_3 \end{bmatrix}$  is in  $W$

8. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If  $w \in \text{Span}\{u, v\}$  and  $T$  is a linear transformation, then  $T(w) \in \text{Span}\{T(u), T(v)\}$

b) For a  $3 \times 3$  matrix  $A$ , if  $A^T = -A$ , then  $\det(A^T A) \leq 0$ .

c) The set  $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_1 \cdot x_2 \geq 0 \right\}$  is a subspace of  $\mathbb{R}^2$ .

a) If  $w \in \text{Span}\{u, v\}$ , then  $w = au + bv$  for some  $a, b \in \mathbb{R}$

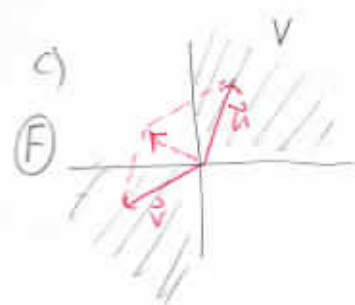
(T) Then  $T(w) = T(au + bv) = T(au) + T(bv) = aT(u) + bT(v)$ , so

$T(w)$  is a lin. comb. of  $T(u)$  and  $T(v)$

b) Let  $A^T = -A$ . Then  $\det(A^T A) = \det A^T \det A = \det(-A) \det A$

$$= (-1)^3 \det A \cdot \det A = -(\det A)^2 \leq 0$$

(T)



Let  $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $v = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ . Then, since  $1 \cdot 2 \geq 0$  and  $(-2)(-1) \geq 0$

$u, v \in V$ . However,  $u+v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ , not in  $V$

since  $(-1) \cdot 1 < 0$ .

**Bonus.** (10pts) Let  $0, A, B$  and  $C$  be  $2 \times 2$  matrices, where  $0$  is a zero matrix. Let  $D$  be the  $4 \times 4$  block matrix below, built using  $0, A, B$  and  $C$ . Show  $\det D = \det A \cdot \det B$ . (Hint: Only write out entries in one of the matrices  $A$  or  $B$  and expand by a row or a column.)

$$D = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \quad \det D = \begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = \begin{matrix} \text{expand} \\ \text{by} \\ \text{row 3} \end{matrix} = e \begin{vmatrix} A & x \\ 0 & h \end{vmatrix} - f \begin{vmatrix} A & x \\ 0 & g \end{vmatrix}$$

$$= e(\det A \cdot h) - f(\det A \cdot g) = \det A (eh - fg) = \det A \det B$$