Matrix Theory — Exam 1
MAT 335, Spring 2026 — D. Ivanšić

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Show all your work!

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) $BA+C$ b) AB c) $2CB+B$

$A = \begin{bmatrix} 4 & -2 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 1 & -3 \\ -2 & -4 & 3 \end{bmatrix}$

$C = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$

a) BA is not defined
$(2\times 3)(1\times 2)$

b) $AB = \begin{bmatrix} 4 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 & -3 \\ -2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 0+4 & 4+8 & -12-6 \end{bmatrix}$
$= \begin{bmatrix} 4 & 12 & -18 \end{bmatrix}$

c) $2\begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -3 \\ -2 & -4 & 3 \end{bmatrix} = 2\begin{bmatrix} 0-4 & 3-8 & -9+6 \\ 0+2 & -1+4 & 3-3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -3 \\ -2 & -4 & 3 \end{bmatrix}$

$= 2\begin{bmatrix} -4 & -5 & -3 \\ 2 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -3 \\ -2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -8 & -9 & -9 \\ 2 & 2 & 3 \end{bmatrix}$

2. (8pts) The solution of a linear system in four variables is given below in vector form.
a) Write a system of equations in usual form (variables left, constants right) that has this solution. (Don't do much - simply reverse the last step in the process of solving a system.)
b) Write the augmented matrix of the system.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \ x_3 \text{ free.}$$

$x_1 = 2+3x_3$
$x_2 = -1-2x_3$
$x_3 = \quad x_3$
$x_4 = 6$

$x_1 \quad - 3x_3 = 2$
$x_2 + 2x_3 = -1$
$x_4 = 6$

b) $\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$

3. (8pts) If A_θ is the 2×2 rotation matrix for a counterclockwise rotation around the origin by angle θ, show that $A_{\frac{\pi}{3}}A_{\frac{\pi}{2}} = A_{\frac{5\pi}{6}}$ using actual numbers in the matrices.

$\begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

$= \begin{bmatrix} \cos\frac{5\pi}{6} & -\sin\frac{5\pi}{6} \\ \sin\frac{5\pi}{6} & \cos\frac{5\pi}{6} \end{bmatrix}$

4. (14pts) A system of linear equations is given below.
a) Use the Gaussian elimination to solve the system.
b) Write the solution in vector form.

$$\begin{cases} x_1 + 3x_2 + 2x_3 - 2x_4 = 11 \\ -x_1 - 3x_2 - x_3 + 2x_4 = -8 \\ 2x_1 + 6x_2 + 3x_3 - 4x_4 = 19 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -2 & 11 \\ -1 & -3 & -1 & 2 & -8 \\ 2 & 6 & 3 & -4 & 19 \end{array}\right] \cdot 1 \quad \cdot(-2)$$

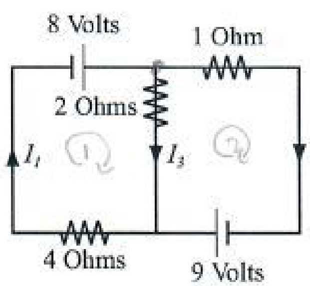
$$\sim \begin{bmatrix} 1 & 3 & 2 & -2 & 11 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 & -3 \end{bmatrix} \cdot 1 \sim \begin{bmatrix} 1 & 3 & 2 & -2 & 11 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix} \cdot(-2) \sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & -2 & 5 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]$$

Free var

$$x_1 + 3x_2 - 2x_4 = 5 \qquad x_1 = 5 - 3x_2 + 2x_4$$
$$x_3 = 3 \qquad x_3 = 3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

5. (10pts) Determine the currents in each branch of the electrical circuit.



1) $4I_1 + 2I_3 = 8$

2) $I_2 - 2I_3 = 9$

$I_1 = I_2 + I_3$

$I_1 - I_2 - I_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 4 & 0 & 2 & 8 \\ 0 & 1 & -2 & 9 \end{array}\right] \cdot(-4)$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 4 & 6 & 8 \\ 0 & 1 & -2 & 9 \end{bmatrix} \cdot \tfrac{1}{2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 9 \\ 0 & 2 & 3 & 4 \end{array}\right] \cdot(-2)$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & 7 & -14 \end{bmatrix} \sim *$$

$$* \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & 1 & -2 \end{bmatrix} \cdot 1 \;\; \cdot 2 \sim \begin{bmatrix} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \cdot 1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \qquad I_1 = 3,\; I_2 = 5,\; I_3 = -2$$

6. (12pts) Below is the augmented matrix of a system of linear equations. Determine the coefficient c for which the system has: a) no solutions, b) one solution, c) infinitely many solutions. (Note: no row operations are needed.)

$$A = \left[\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & c+4 & -2 & 0 \\ 0 & 0 & c^2+2c-8 & c-2 \end{array}\right]$$

b) one solution if

$c^2+2c-8 \neq 0$ and $c+4 \neq 0$

$(c+4)(c-2) \neq 0$ $\quad c \neq -4,\ c \neq 2$

a) no solution if

$c^2+2c-8=0$ and $c-2 \neq 0$

$(c+4)(c-2)=0$ and $c-2 \neq 0$

$\Rightarrow c+4=0,\ c=-4$

c) infinitely solutions if

$c^2+2c-8=0$ and $c-2=0$ or $\cancel{c+4=0}$

$(c+4)(c-2)=0$ $\quad c-2=0$ $\quad$ gives no sol

$c=2$

7. (6pts) Find the elementary matrix E so that $EA = B$.

$$A = \begin{bmatrix} 0 & -2 & 7 \\ 3 & 2 & -1 \\ -4 & 6 & 0 \end{bmatrix} \cdot 2, \quad B = \begin{bmatrix} 0 & -2 & 7 \\ -5 & 14 & -1 \\ -4 & 6 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

8. (12pts) Consider the vectors $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 8 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$.

a) Do they span $\mathbf{R}^3$?

b) If not, which of the vectors can be removed so that the remaining two have the same span as the original three?

$$\begin{array}{c} \\ (-3) \\ \cdot 1 \end{array} \begin{bmatrix} 1 & -6 & 5 \\ 3 & 8 & 2 \\ -1 & -6 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 5 \\ 0 & 26 & -13 \\ 0 & -12 & 6 \end{bmatrix} \begin{array}{c} \\ \cdot\frac{1}{13} \\ (-\frac{1}{6}) \end{array} \sim \begin{bmatrix} 1 & -6 & 5 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 5 \\ 0 & 2 & -1 \end{bmatrix} \cdot (+3)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

rank = 2, they don't span $\mathbb{R}^3$

$\vec{r}_3 = 2\vec{r}_1 - \frac{1}{2}\vec{r}_2$, so $\vec{a}_3 = 2\vec{a}_1 - \frac{1}{2}\vec{a}_2$

so we can remove $\vec{a}_3 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$

9. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If **a** is in Span{**u**, **v**}, then **v** is in Span{**u**, **a**}.
b) A 2×3 matrix can have rank equal to 0, 1, 2, or 3.
c) For 2×2 matrices A and B, where A is diagonal, $AB = BA$.

a) $\vec{a} = c_1\vec{u} + c_2\vec{v}$

$c_2\vec{v} = \vec{a} - c_1\vec{u}$

$\vec{v} = \frac{1}{c_2}\vec{a} - \frac{c_1}{c_2}\vec{u}$

works only if $c_2 \neq 0$

hints at counterexample F

$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{a}$ is in Span$\{\vec{u}, \vec{v}\}$

but $\vec{v}$ is not in Span$\{\vec{u}, \vec{a}\}$

as its 2nd coord is 0

b) F rank $A \leq \min\{\text{no. of rows}, \text{no. of colum}\}$

so rank $A \leq \min\{2, 3\} = 2$

rank $A \leq 2$

c) F

$$\begin{array}{cc} A & B \end{array}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

not equal

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\begin{array}{cc} B & A \end{array}$$

Bonus. (10pts) Show: if $c \neq 0$, then Span{**u**, **v**} = Span{**u**, **u** + c**v**}.

Let $c \neq 0$, then $\vec{v} = \frac{1}{c} c\vec{v} = \frac{1}{c}(\vec{u} + c\vec{v} - \vec{u}) = \frac{1}{c}(\vec{u} + c\vec{v}) - \frac{1}{c}\vec{u}$

($\vec{v}$ is a lin. comb. of $\vec{u} + c\vec{v}$ and $\vec{u}$)

Thus Span$\{\vec{u}, \vec{v}\}$ = Span$\{\vec{u}, \vec{v}, \vec{u} + c\vec{v}\}$ = Span$\{\vec{u}, \vec{u} + c\vec{v}\}$

since $\vec{u} + c\vec{v}$ is a lin comb of $\vec{u}, \vec{v}$

since $\vec{v}$ is a lin comb of $\vec{u} + c\vec{v}$, $\vec{u}$