Matrix Theory — Lecture notes **MAT 335, Spring 2026 — D. Ivanšić**

5.1 Eigenvalues and Eigenvectors

Linear transformations $T : \mathbf{R}^n \to \mathbf{R}^n$ are the most often studied ones, so they get a special name: *linear operators.*

Example. Matrices of linear operators $\mathbf{R}^n \to \mathbf{R}^n$ often obscure the simple action of the linear operator. For example, reflections in lines in $\mathbf{R}^2$ or planes in $\mathbf{R}^3$, or rotations around an axis in $\mathbf{R}^3$ usually have complicated matrices when the lines, planes or axes in question are not aligned with standard basis vectors. How to find what lines or planes we are reflecting or rotating about?

Definition. Let T be a linear operator on $\mathbf{R}^n$. A nonzero vector $\mathbf{v}$ in $\mathbf{R}^n$ is called an *eigenvector* of T if $T(\mathbf{v}) = \lambda\mathbf{v}$ for some scalar λ (in other words, if in direction of $\mathbf{v}$, T acts simply by dilating $\mathbf{v}$). The scalar λ is called the *eigenvalue* of T that corresponds to $\mathbf{v}$.

Definition. Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{v}$ in $\mathbf{R}^n$ is called an *eigenvector* of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. The scalar λ is called the *eigenvalue* of A that corresponds to $\mathbf{v}$.

Note. If $\mathbf{v}$ is an eigenvector, then so is any multiple $c\mathbf{v}$ of $\mathbf{v}$. An eigenvector cannot correspond to different eigenvalues.

Proposition. The eigenvectors and corresponding eigenvalues of a linear operator are the same as those of its standard matrix.

Proposition. If λ is an eigenvalue of A or T, then

— the eigenvectors corresponding to λ are the nonzero solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$,

— the eigenvectors and $\mathbf{0}$ form a subspace of $\mathbf{R}^n$ called the *eigenspace* of A (or T).

Proof.

Example. Show that 2 is an eigenvalue of the matrix below and find a basis for its eigenspace.

$$A = \begin{bmatrix} 15 & -13 & 26 \\ 5 & -3 & 10 \\ -3 & 3 & -4 \end{bmatrix}$$

Example. Show that the matrix below represents a reflection in a line, and state a vector on the line.

$$A = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}$$

5.2 The Characteristic Polynomial

Proposition. The number λ is an eigenvalue of a square matrix A if and only if $\det(A - \lambda I) = 0$.

Proof.

Example. Determine the eigenvalues of the matrix below and find a basis for each corresponding eigenspace.

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 8 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Definition. For an $n \times n$ matrix A, the function $\det(A - tI)$ is a degree n polynomial, called the *characteristic polynomial* of A. The equation $\det(A - tI) = 0$ is called the *characteristic equation* of A, and its real roots are the eigenvalues of A.

Theorem 5.1. Let λ be an eigenvalue of a square matrix A. The dimension of the eigenspace of A corresponding to λ is less than or equal to the multiplicity of λ in the characteristic polynomial of A.

Note. The eigenvalues of an upper or lower triangular matrix are the diagonal entries.

Note. The matrix and its reduced row echelon form do not, in general, have the same characteristic polynomial — check previous example.

Example. Some matrices (linear operators) do not have any eigenvalues. Show this is the case in an algebraic and geometric way for most rotations.