

Definition. A nonempty set W of vectors in \mathbf{R}^n is called a *subspace* of \mathbf{R}^n if it has these two properties:

1. If \mathbf{u} and \mathbf{v} are in W , then $\mathbf{u} + \mathbf{v}$ is in W . (We say W is *closed under vector addition*.)
2. If \mathbf{u} is in W , then $c\mathbf{u}$ is in W for any scalar c . (We say W is *closed under scalar multiplication*.)

Proposition. If W is a subspace of \mathbf{R}^n , then $\mathbf{0}$ is in W .

Proof.

Example. Are the following sets subspaces of \mathbf{R}^3 ?

$$W_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid 2x_1 + x_2 - 4x_3 = 1 \right\} \quad W_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid 2x_1 + x_2 - 4x_3 = 0 \right\}$$

Example. The one-element set $\{\mathbf{0}\}$ is a subspace of \mathbf{R}^n . The set \mathbf{R}^n is a subspace of \mathbf{R}^n .

Example. Let W be the set of all vectors in \mathbf{R}^2 whose both components are integers. Is W a subspace of \mathbf{R}^2 ?

Example (Theorem 4.1). If \mathcal{S} is any nonempty subset of \mathbf{R}^n , then $\text{Span } \mathcal{S}$ is a subspace of \mathbf{R}^n . With this in mind, draw some subspaces of \mathbf{R}^3 .

Definition (subspaces associated with a matrix). Let A be an $m \times n$ matrix.

- The *null space* of A is the solution set of $A\mathbf{x} = \mathbf{0}$, denoted by $\text{Null } A$.
- The *column space* of A is the span of the columns of A , denoted by $\text{Col } A$.
- The *row space* of A is the span of the rows of A , denoted by $\text{Row } A$.

Theorem 4.2. Let A be an $m \times n$ matrix.

- The null space of A is a subspace of \mathbf{R}^n .
- The column space of A is a subspace of \mathbf{R}^m .
- The row space of A is a subspace of \mathbf{R}^n .

Proof.

Example. Write the generating sets for the null space, the column space and row space for the matrix below.

$$A = \begin{bmatrix} 1 & -4 & 7 & 1 & 3 \\ 3 & -10 & 27 & 2 & -1 \\ 1 & -3 & 10 & -1 & 5 \\ 2 & -8 & 14 & 4 & -3 \end{bmatrix}$$

Subspaces associated with a linear transformation. Let $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a linear transformation with standard matrix A . Recall that the null space of T is all vectors x so that $T(\mathbf{x}) = \mathbf{0}$.

- The null space of T is the same as the null space of A , so it is a subspace of \mathbf{R}^n .
- The range of T is the column space of A , so it is a subspace of \mathbf{R}^m .

Definition. Let V be a nonzero subspace of \mathbf{R}^n . A *basis* for V is a linearly independent generating set.

Example.

$\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for \mathbf{R}^n Any two nonparallel vectors in \mathbf{R}^2 are a basis for \mathbf{R}^2

Recall these theorems:

Theorem 1.7. $\text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}\} = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ if and only if \mathbf{v} is in $\text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$.

Theorem 1.9. $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly dependent if and only if $\mathbf{u}_1 = \mathbf{0}$ or there is an $i \geq 2$ such that \mathbf{u}_i is a linear combination of its predecessors.

Theorem 4.3 (Reduction Theorem). If \mathcal{S} is a finite generating set for a nonzero subspace V or \mathbf{R}^n , then \mathcal{S} can be reduced to a basis for V by removing vectors from \mathcal{S} .

Proof.

Example. Reduce the following generating set to a basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 11 \end{bmatrix} \right\}$

Theorem 4.4 (Extension Theorem). Let \mathcal{S} be a linearly independent subset of a nonzero subspace V of \mathbf{R}^n . Then \mathcal{S} can be extended to a basis for V by including additional vectors from V in \mathcal{S} . In particular, every nonzero subspace has a basis, which means every subspace is a span of finitely many vectors.

Proof.

Theorem 4.5. Let V be a subspace of \mathbf{R}^n . Then any two bases for V contain the same number of elements.

Proof.

Definition. The number of vectors in a basis for a nonzero subspace V or \mathbf{R}^n is called the *dimension* of V , denoted $\dim V$. We define the dimension of the zero subspace to be 0.

Example. Is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 3 \\ 0 \end{bmatrix} \right\}$ a basis for \mathbf{R}^4 ?

Theorem 4.6. Let V be a k -dimensional subspace of \mathbf{R}^n . Then

- Every linearly independent set of V contains at most k vectors.
- Every subset with more than k vectors is linearly dependent.

Proof.

Theorem 4.7. Let V be a k -dimensional subspace of \mathbf{R}^n , and \mathcal{S} a set with k vectors. Then

- If \mathcal{S} is linearly independent, it is a basis for V .
- If \mathcal{S} is a generating set for V , it is a basis for V .

Proof.

Example. Find a basis for the null space and column space of the matrix below.

$$A = \begin{bmatrix} -3 & -6 & 3 & -2 \\ -2 & -5 & 3 & 5 \\ 4 & 9 & -5 & 6 \end{bmatrix}$$

Example.

Is $\left\{ \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 21 \\ -17 \end{bmatrix} \right\}$ a basis for the subspace generated by $\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$?

Summary. To show a set \mathcal{B} is a basis for a subspace V of \mathbf{R}^n :

1. Show \mathcal{B} is contained in V .
2. Show \mathcal{B} is linearly independent or that it generates V .
3. Compute $\dim V$ and check it equals the number of vectors in \mathcal{B} .

Proposition. Let A be an $m \times n$ matrix. The following are dimensions of subspaces associated with the matrix A .

Subspace	Notation	Subspace of	Dimension
column space of A	$\text{Col } A$	\mathbf{R}^m	$\text{rank } A$
null space of A	$\text{Null } A$	\mathbf{R}^n	$\text{nullity } A = n - \text{rank } A$
row space of A	$\text{Row } A$	\mathbf{R}^n	$\text{rank } A$

Proof.

Theorem 4.8. The nonzero rows in the reduced row echelon form of a matrix are the basis of the row space of the matrix.

Proof. We show that $\text{Row } A = \text{Row } R$.

Note. $\text{rank } A = \text{rank } A^T$

Theorem 4.9. If V and W are subspaces of \mathbf{R}^n and V is contained in W , then $\dim V \leq \dim W$. If, additionally, $\dim V = \dim W$, then $V = W$.