

Example. Find a simple condition under which the system below has a solution for every $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Because the corresponding matrix A is 2×2 , having a solution for every \mathbf{b} automatically guarantees it is unique, because it boils down to $\text{rank } A = 2$, which is further equivalent to matrix A being invertible.

$$\begin{cases} ax_1 + bx_2 = b_1 \\ cx_1 + dx_2 = b_2 \end{cases}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Proposition. The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$.

Because the scalar $ad - bc$ *determines* whether A is invertible, it is called the *determinant* of A and denoted $\det A$ (or $|A|$, when the matrix is written out). For square matrices, the determinant is defined recursively, by “expansion along a row”.

Example. Find the determinant of the matrix below by expanding it
a) along the first row b) along the second row.

$$\begin{vmatrix} 7 & 1 & 1 \\ 4 & -2 & 0 \\ -1 & 3 & -4 \end{vmatrix}$$

Definition. Let A be an $n \times n$ matrix, and let A_{ij} be the $(n - 1) \times (n - 1)$ matrix obtained by removing the i -th row and j -th column of A . The *determinant* of A by expansion along row i is the number

$$\det A = a_{i1}(-1)^{i+1} \det A_{i1} + a_{i2}(-1)^{i+2} \det A_{i2} + \cdots + a_{ij}(-1)^{i+j} \det A_{ij} + \cdots + a_{in}(-1)^{i+n} \det A_{in}$$

If we set $c_{ij} = (-1)^{i+j} \det A_{ij}$, and call c_{ij} the (i, j) -*cofactor* of A , we can write this as

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

and call it the cofactor expansion of A along row i .

Theorem 3.1. For all $i = 1, \dots, n$, the determinant of A by expansion along row i results in the same number.

Note. The determinant of a 2×2 matrix is essentially expansion along the first row, if we understand the determinant of a 1×1 matrix to be the single scalar it contains.

Example. Find the determinant of the matrix below.

$$\begin{vmatrix} 4 & 7 & 3 & 0 & 2 \\ 2 & -9 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 4 & -1 & 1 & 2 \\ 4 & -3 & 0 & 5 & 0 \end{vmatrix} =$$

Note.

To compute a	we have
2×2 determinant	2 multiplications, 1 addition
3×3 determinant	$> 3 \cdot 2$ multiplications, $3 \cdot 1 + (3 - 1) = 5 = 3! - 1$ additions
4×4 determinant	$> 4 \cdot 3 \cdot 2$ multiplications, $4(3! - 1) + 3 = 4! - 1$ additions
5×5 determinant	$> 5 \cdot 4 \cdot 3 \cdot 2$ multiplications, $5(4! - 1) + 4 = 5! - 1$ additions
\vdots	\vdots
$n \times n$ determinant	$> n!$ multiplications, $n! - 1$ additions

For a smallish 20×20 determinant, this is about 4.87×10^{18} multiplications and additions. If each operation takes 10^{-9} seconds on a computer, this takes some 154 years — impractical! This is not the way to compute large determinants.

Definition. An $n \times n$ matrix is

- *upper triangular*, if all entries below the diagonal are 0
- *lower triangular*, if all entries above the diagonal are 0

Example. A is upper and B is lower triangular. Find $\det A$.

$$A = \begin{bmatrix} 7 & 1 & 1 & -3 \\ 0 & -2 & 0 & 5 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -1 & 2 & -4 & 0 \\ 0 & 1 & 7 & 5 \end{bmatrix}$$

Theorem 3.2. The determinant of an upper or lower triangular matrix is the product of the diagonal entries. In particular, $\det I_n = 1$.

Proposition. Let \mathbf{a}, \mathbf{b} be vectors in \mathbf{R}^2 , and $\mathbf{c}, \mathbf{d}, \mathbf{e}$ vectors in \mathbf{R}^3 .

- The area of a parallelogram spanned by \mathbf{a}, \mathbf{b} is $|\det [\mathbf{a} \ \mathbf{b}]|$.
- The volume of the parallelepiped spanned by $\mathbf{c}, \mathbf{d}, \mathbf{e}$ is $|\det [\mathbf{c} \ \mathbf{d} \ \mathbf{e}]|$.

Example. Find the area of the triangle ABC , if $A = (1, 3)$, $B = (3, 0)$ and $C = (2, -1)$.

Note. Getting zero for the determinants above tells us \mathbf{a} and \mathbf{b} are parallel, and vectors \mathbf{c}, \mathbf{d} and \mathbf{e} are in the same plane. In other words, $\{\mathbf{a}, \mathbf{b}\}$ does not span \mathbf{R}^2 , and $\{\mathbf{c}, \mathbf{d}, \mathbf{e}\}$ does not span \mathbf{R}^3 , which we already knew, because the matrices involved are not invertible.

Theorem 3.4. Let A and B $n \times n$ matrices. The following statements are true:

- (a) A is invertible if and only if $\det A \neq 0$.
- (b) $\det(AB) = \det A \cdot \det B$
- (c) $\det A^T = \det A$ (This implies we can expand determinants by columns.)
- (d) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$

Proof. Statement b) is harder to prove, and we omit it. Assuming b), we can easily show a), d) and c).

Proposition. Let A be an invertible $n \times n$ matrix, and let C be the *cofactor matrix* of A : its (i, j) -entry is the (i, j) -cofactor of the matrix A . Then

$$A^{-1} = \frac{1}{\det A} C^T \quad \text{In particular,} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 3.3.

Let A be an $n \times n$ matrix.

If B is obtained from A by:	Then:
swapping two rows of A	$\det B = -\det A$
multiplying a row of A by k	$\det B = k \det A$
adding a multiple of a row to another row	$\det B = \det A$

Proof. Since $B = EA$, for some elementary matrix E , so $\det B = \det E \cdot \det A$, we just need to find determinants of elementary matrices.

Proposition. If a square matrix has two proportional rows, its determinant is 0.

Example. Use row operations to find the determinant of the matrix below, a much more efficient method than expansion by rows for large matrices.

$$\begin{vmatrix} 4 & 3 & 0 & 2 \\ 2 & 1 & -2 & 3 \\ 2 & -1 & 1 & 2 \\ 4 & 0 & 5 & 2 \end{vmatrix} =$$

Theorem 3.5 (Cramer's Rule). Let A be an invertible $n \times n$ matrix, \mathbf{x} in \mathbf{R}^n and let M_i be the matrix obtained from A by replacing column i of A by \mathbf{b} . Then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution \mathbf{x} with components

$$x_i = \frac{\det M_i}{\det A} \text{ for } i = 1, \dots, n.$$

Proof.

Example. Solve the system below with Cramer's rule.

$$\begin{cases} 2x_1 - 3x_2 & = 1 \\ x_1 + 2x_2 + 3x_3 & = 4 \\ -4x_1 + 2x_2 - 5x_3 & = 0 \end{cases}$$