Calculus 2 — Final Exam
MAT 308, Spring 2026 — D. Ivanšić

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Show all your work!

Find the following integrals:

1. (6pts) $\int x \ln x \, dx = \begin{bmatrix} u = \ln x & dv = x\,dx \\ du = \frac{1}{x}dx & v = \frac{x^2}{2} \end{bmatrix} = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$

$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

2. (10pts) $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \int_0^{\pi/2} \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int_0^{\pi/2} 1 + 2\cos(2x) + \cos^2(2x)\,dx$

$= \frac{1}{4}\int_0^{\pi/2} 1 + 2\cos(2x) + \frac{1+\cos(4x)}{2}\,dx = \frac{1}{4}\int_0^{\pi/2} \frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x)\,dx$

$= \frac{1}{4}\left(\frac{3}{2}\cdot\frac{\pi}{2} + \sin(2x)\Big|_0^{\pi/2} + \frac{1}{8}\sin(4x)\Big|_0^{\pi/2}\right) = \frac{3\pi}{16}$

$\sin\pi - \sin 0 = 0$ $\quad$ $\sin 2\pi - \sin 0 = 0$

3. (12pts) Determine whether the following improper integral converges by calculating it directly.

$\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx = \begin{bmatrix} u = x & dv = e^{-x}dx \\ du = 1\,dx & v = -e^{-x} \end{bmatrix} = \lim_{t\to\infty}\left(-xe^{-x}\Big|_0^t + \int_0^t e^{-x}\,dx\right)$

$= \lim_{t\to\infty}\left(-te^{-t} - 0 + (-e^{-x})\Big|_0^t\right) = \lim_{t\to\infty}\left(-te^{-t} - (e^{-t} - e^0)\right)$

$= \lim_{t\to\infty}\left(-\frac{t}{e^t} - \frac{1}{e^t} + 1\right) = \lim_{t\to\infty} -\frac{1}{e^t} - 0 + 1 = 0 + 1 = 1$

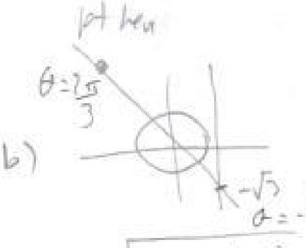
$\uparrow$
$\frac{\infty}{\infty}$ use L'H

The integral converges

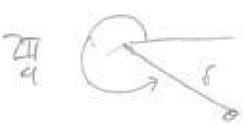
4. (10pts) Convert (a picture may help):

a) $\left(8, \dfrac{7\pi}{4}\right)$ from polar to rectangular coordinates

b) $(-2\sqrt{3}, 6)$ from rectangular to polar coordinates



a)



$x = 8\cos\frac{7\pi}{4} = 8\frac{\sqrt{2}}{2} = 4\sqrt{2}$

$y = 8\sin\frac{7\pi}{4} = 8\left(-\frac{\sqrt{2}}{2}\right) = -4\sqrt{2}$

$r = \sqrt{(-2\sqrt{3})^2 + 6^2} = \sqrt{12+36} = \sqrt{48} = 4\sqrt{3}$

$\tan\theta = \dfrac{6}{-2\sqrt{3}} = -\dfrac{3}{\sqrt{3}} = -\sqrt{3}, \quad \theta = \dfrac{2\pi}{3}$

$\left(4\sqrt{3}, \dfrac{2\pi}{3}\right)$

5. (24pts) The region bounded by the curves $y = 2 + \sqrt{x}$ and $y = \dfrac{x}{2} + 2$ is rotated around the x-axis.

a) Sketch the solid and a typical cross-sectional washer.

b) Set up the integral for the volume of the solid.

c) On another picture, sketch the solid and a typical cylindrical shell.

d) Set up the integral for the volume of the solid using the shell method.

Simplify, but do not evaluate the integrals.

$A = \pi r_2^2 - \pi r_1^2$

$2 + \sqrt{x} = \frac{x}{2} + 2$

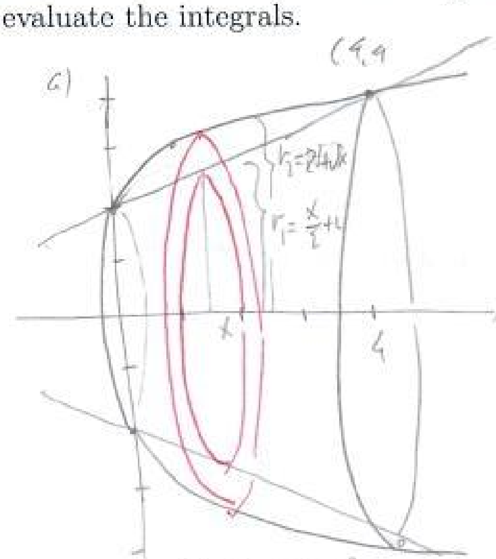
$\sqrt{x} = \frac{x}{2}$

$x - 2\sqrt{x} = 0$

$\sqrt{x}(\sqrt{x} - 2) = 0$

$\sqrt{x} = 0 \quad \sqrt{x} = 2$

$x = 0 \quad x = 4$



a) $V = \displaystyle\int_0^4 \pi\left(\left(2+\sqrt{x}\right)^2 - \left(2+\frac{x}{2}\right)^2\right)dx$

$= \displaystyle\int_0^4 \pi\left(4 + 4\sqrt{x} + x - \left(4 + 2x + \frac{x^2}{4}\right)\right)dx$

$= \displaystyle\int_0^4 \pi\left(4\sqrt{x} - x - \frac{x^2}{4}\right)dx$

$y = \frac{x}{2} + 2$

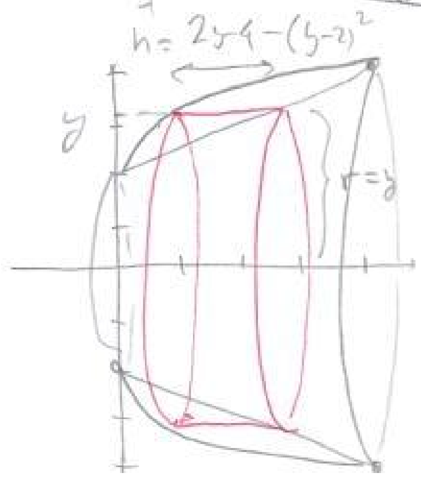
$2y = x + 4$

$x = 2y - 4$

$y = 2 + \sqrt{x}$

$\sqrt{x} = y - 2$

$x = (y-2)^2$



b) $A = 2\pi r h$

$V = \displaystyle\int_2^4 2\pi y\left(2y - 4 - (y-2)^2\right)dy$

$= \displaystyle\int_2^4 2\pi y\left(2y - 4 - (y^2 - 4y + 4)\right)dy$

$= \displaystyle\int_2^4 2\pi y\left(-y^2 + 6y - 8\right)dy$

6. (10pts) Justify why the series converges and find its sum.

$$\sum_{n=2}^{\infty}(-1)^n\frac{6\cdot 7^n}{2^{3n+2}} = \sum_{n=2}^{\infty} 6(-1)^n \frac{7^n}{(2^3)^n\cdot 2^2} = \sum_{n=2}^{\infty}\frac{6}{4}\left(-\frac{7}{8}\right)^n = \sum_{n=2}^{\infty}\frac{3}{2}\left(-\frac{7}{8}\right)^n$$

geometric series, $r=-\frac{7}{8}$

$|r|<1$

$$= \frac{\frac{3}{2}\cdot\left(-\frac{7}{8}\right)^2}{1-\left(-\frac{7}{8}\right)} = \frac{\frac{3\cdot 49}{2\cdot 64}}{1+\frac{7}{8}} = \frac{\frac{3\cdot 49}{128}}{\frac{15}{8}} = \frac{\cancel{3}^{1}\cdot 49}{\cancel{128}_{16}}\cdot\frac{\cancel{8}^{1}}{\cancel{15}_{5}} = \frac{49}{80}$$

7. (14pts) Find the interval of convergence of the series. Don't forget to check the endpoints.

$$\sum_{n=1}^{\infty}\frac{2^{n+2}(x-4)^n}{\sqrt{n}}$$

$$\sqrt[n]{\left|\frac{2^{n+2}(x-4)^n}{\sqrt{n}}\right|} = \sqrt[n]{\frac{2^n|x-4|^n\cdot 2^2}{\sqrt{n}}} = \frac{2|x-4|\sqrt[n]{4}}{\sqrt[n]{\sqrt{n}}} \to \frac{2|x-4|\cdot 1}{\sqrt{1}}$$

$\sqrt[n]{\sqrt{n}} = \sqrt{\sqrt[n]{n}}$

Converges if $2|x-4|<1$

$|x-4|<\frac{1}{2}$

$-\frac{1}{2}<x-4<\frac{1}{2}$

$3\frac{1}{2}<x<4\frac{1}{2}$

When $x=4\frac{1}{2}$ get $\sum_{n=1}^{\infty}\frac{2^{n+2}\left(\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{4}{\sqrt{n}} = 4\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges, p-series $p<1$ $\sum\frac{1}{n^p}$

When $x=3\frac{1}{2}$ get $\sum_{n=1}^{\infty}\frac{2^{n+2}\left(-\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n 4}{\sqrt{n}}$ $\quad \frac{4}{\sqrt{n}}$ decreasing $\lim_{n\to\infty}\frac{4}{\sqrt{n}}=0$

Converges by alternating series test

Converges on $\left[3\frac{1}{2}, 4\frac{1}{2}\right)$

or $\left[\frac{7}{2}, \frac{9}{2}\right)$

8. (18pts) Let $f(x) = \sqrt[3]{x}$.

a) Find the 3rd Taylor polynomial for f centered at $a = 8$.

b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $(6, 10)$.

n	$f^{(n)}$	$f^{(n)}(8)$
0	$x^{1/3}$	2
1	$\frac{1}{3}x^{-\frac{2}{3}}$	$\frac{1}{3}\cdot\frac{1}{2^2} = \frac{1}{12}$
2	$-\frac{2}{9}x^{-\frac{5}{3}}$	$-\frac{2}{9}\cdot\frac{1}{2^5} = \frac{-1}{144}$
3	$\frac{10}{27}x^{-\frac{8}{3}}$	$\frac{10}{27}\cdot\frac{1}{2^8} = \frac{5}{27\cdot 128} = \frac{5}{3456}$
4	$-\frac{80}{81}x^{-\frac{11}{3}}$	

$$T_3(x) = 2 + \frac{1}{12}(x-2) - \frac{\frac{1}{144}}{2}(x-2)^2 + \frac{\frac{5}{3456}}{6}(x-2)^3$$

$$T_3(x) = 2 + \frac{1}{12}(x-2) - \frac{1}{288}(x-2)^2 + \frac{5}{20736}(x-2)^3$$

$$8^{-\frac{k}{3}} = (8^{\frac{1}{3}})^{-k} = 2^{-k} = \frac{1}{2^k}$$

$\frac{80}{81}x^{-\frac{11}{3}}$ decr, biggest at $x=6$

$6 < x < 10$

$-2 < x-8 < 2$

$|x-8| < 2$

$$|R_3(x)| \le \left|\frac{f^{(4)}(2)}{4!}(x-8)^4\right| \le$$

$$\le \frac{\frac{80}{81}6^{-\frac{11}{3}}\,2^4}{24} = \frac{2^4\cdot \cancel{80}^{\,20}}{81\cdot \cancel{24}_{\,6}\cdot 6^{11/3}}$$

$$\le \frac{2^4\cdot 20}{81\cdot 6^3} = \frac{2^{\cancel{4}\,3}\cdot \cancel{20}^{\,5}}{81\cdot \cancel{6}_{9}\cdot\cancel{6}_{3}\cdot 6} = \frac{40}{81\cdot 27}$$

$6^{11/3} > 6^{9/3} = 6^3$

$81\cdot 9\cdot 3$

$729\cdot 3$

$$= \frac{40}{2187}$$

$128\cdot 27$ / 896 / 256 / 3456

$3456\cdot 6$ / 20736

9. (10pts) A particle moves along the path with parametric equations $x(t) = e^t + 1$, $y(t) = e^{2t} - 1$, t any. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

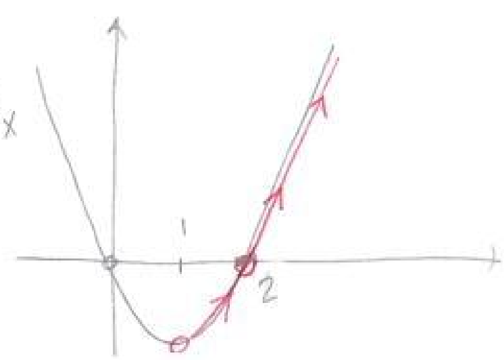
$x = e^t + 1$ so $e^t = x - 1$

$y = e^{2t} - 1 = (e^t)^2 - 1 = (x-1)^2 - 1 = x^2 - 2x + 1 - 1 = x^2 - 2x$

Since $e^t > 0$, $x = e^t + 1 > 1$

x increasing with t

$y = x^2 - x$



10. (24pts) The integral $\int_0^1 e^{-x^2}\,dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of $f(x) = e^{-x^2}$ is not expressible using elementary functions.
a) Write the expression you would use to calculate M_6, the midpoint rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) It is known that $-2 < f''(x) < \frac{4}{5}$ on $[0,1]$: use it to find the error estimate for M_n in general.
c) What should n be in order for M_n to give you an error less than 10^{-4}?
d) Use the known power series for e^x to find a power series for the above integral.
e) How many terms of the power series are needed to estimate the integral to accuracy 10^{-4}? Write the estimate as a sum (you do not have to simplify it).
f) Which method requires less computation to evaluate the integral with accuracy 10^{-4}, Simpson rule or series?

a) $0 \quad \frac{1}{6} \quad \frac{2}{3} \quad \frac{1}{2} \quad \frac{4}{3} \quad \frac{5}{6} \quad 1$

$\frac{1}{12} \quad \frac{1}{4} \quad \frac{5}{12} \quad \frac{7}{12} \quad \frac{3}{4} \quad \frac{11}{12}$

$$M_6 = \frac{1}{6}\left(e^{-\frac{1}{12^2}} + e^{-\frac{1}{4^2}} + e^{-\frac{5^2}{12^2}} + e^{-\frac{7^2}{12^2}} + e^{-\frac{3^2}{4^2}} + e^{-\frac{11^2}{12^2}}\right)$$

$|f''(x)| \le 2$

$\Delta x = \frac{1}{6}$

b) $$\left|\int_0^1 e^{-x^2}dx - M_6\right| < \frac{K_2(b-a)^3}{24n^2} \le \frac{2\cdot(1-0)^3}{24n^2} = \frac{1}{12n^2}$$

c) $$\frac{1}{12n^2} < 10^{-4} \quad 12n^2 > 10^4, \quad n^2 > \frac{10^4}{12}, \quad n > \frac{10^2}{2\sqrt{3}} = \frac{100}{2\sqrt{3}} = \frac{50}{\sqrt{3}} \text{ about } 30$$

d) $$\int_0^1 e^{-x^2}dx = \int_0^1 \sum_{n=0}^\infty \frac{(-x^2)^n}{n!}dx = \sum_{n=0}^\infty \int_0^1 \frac{(-1)^n x^{2n}}{n!}dx = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1}\Big|_0^1 = \sum_{n=0}^\infty \frac{(-1)^n}{n!(2n+1)}$$

Need $\frac{1}{n!(2n+1)} < 10^{-4}$

$(2n+1)n! > 10^4$

n	$(2n+1)\,n!$
4	$9\cdot 4! = 9\cdot 24$
5	$11\cdot 120$
6	$13\cdot 720$
7	$15\cdot 7! > 10000$ $15\cdot 5040$

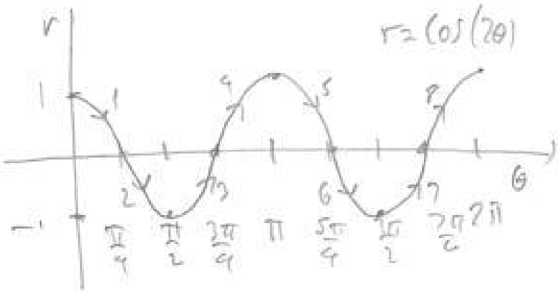
e) Go up to $n=6$

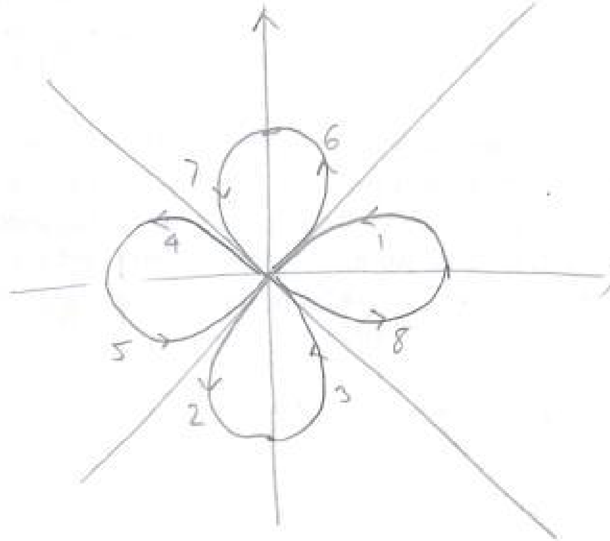
$$S_6 = 1 - \frac{1}{3\cdot 1} + \frac{1}{5\cdot 2} - \frac{1}{7\cdot 6} + \frac{1}{9\cdot 24} - \frac{1}{11\cdot 120} + \frac{1}{13\cdot 720}$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} - \frac{1}{216} - \frac{1}{1320} + \frac{1}{9360}$$

$$\begin{array}{r} 720\cdot 13 \\ \hline 2160 \\ 720 \\ \hline 9360 \end{array}$$

11. (12pts) First draw the graph of $r = \cos(2\theta)$ in a cartesian θ-r coordinates. Use this graph to draw the polar curve with the same equation.





Bonus (15pts) Show the reduction formula.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \sin^n x\,dx = \left[\begin{array}{ll} u=\sin^{n-1}x & dv=\sin x\,dx \\ du=(n-1)\sin^{n-2}x\cos x & v=-\cos x\,dx \end{array}\right] = -\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

$$= -\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$\int \sin^n x\,dx = -\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\,dx-(n-1)\int\sin^n x\,dx \qquad \Big|\ +(n-1)\int \sin^n x\,dx$$

$$n\int \sin^n x\,dx = -\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\,dx$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x+\frac{n-1}{n}\int \sin^{n-2}x\,dx$$