Calculus 2 — Exam 4
MAT 308, Spring 2026 — D. Ivanšić

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Show all your work!

Find the intervals of convergence for the series below. Don't forget to check the endpoints.

1. (10pts) $\sum_{n=1}^{\infty}(-1)^n \frac{10^n}{(2n)!}x^n$

Ratio test $\left|\dfrac{(-1)^{n+1}\frac{10^{n+1}}{(2n+2)!}x^{n+1}}{(-1)^n\frac{10^n}{(2n)!}x^n}\right| = \dfrac{10^{n+1}(2n)!}{\cancel{10^n}(2n+2)!}|x| = \dfrac{10(2n)!}{(2n)!(2n+1)(2n+2)}|x|$

$= \dfrac{10|x|}{(2n+1)(2n+2)} \to \dfrac{10|x|}{\infty} = 0 < 1$ Converges for every x

2. (16pts) $\sum_{n=1}^{\infty}\frac{1}{n3^{n+1}}(x-2)^n$

Root test $\sqrt[n]{\left|\frac{1}{n3^{n+1}}(x-2)^n\right|} = \sqrt[n]{\dfrac{|x-2|^n}{n\cdot 3^n\cdot 3}} = \dfrac{|x-2|}{\sqrt[n]{n}\cdot 3\cdot\sqrt[n]{3}} \to \dfrac{|x-2|}{1\cdot 3\cdot 1} = \dfrac{|x-2|}{3}$

Converges when $\frac{|x-2|}{3} < 1$ $\quad |x-2| < 3 \quad -3 < x-2 < 3 \quad -1 < x < 5$

When $x=-1$ get $\sum_{n=1}^{\infty}\frac{1}{n3^{n+1}}(-3)^n = \sum_{n=1}^{\infty}(-1)^n\frac{3^n}{n3^{n+1}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{3n}$ $\quad \frac{1}{3n}\to 0$ decreasing, converges by alt. series test

When $x=5$ get $\sum_{n=1}^{\infty}\frac{1}{n3^{n+1}}3^n = \sum_{n=1}^{\infty}\frac{3^n}{n3^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, p-series, $p=1$

Interval of convergence $= [-1, 5)$

3. (6pts) Use a known power series to find the sum.

$$\sum_{n=0}^{\infty} \frac{(\ln 2)^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} \cdot \frac{1}{\ln 2} = \frac{1}{\ln 2} \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} = \frac{1}{\ln 2} \cdot e^{\ln 2} = \frac{2}{\ln 2}$$

4. (8pts) Use known power series to find the limit.

$$\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) - x\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right)}{x^3}$$

$$\frac{1}{120} - \frac{1}{24} = -\frac{4}{120} = -\frac{1}{30}$$

$$= \lim_{x\to 0} \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - x + \frac{x^3}{2} - \frac{x^5}{24} + \dots}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{3} - \frac{x^5}{30} + \dots}{x^3}$$

$$= \lim_{x\to 0} \left(\frac{1}{3} - \frac{x^2}{30} + \dots\right) = \frac{1}{3}$$

5. (14pts) Use geometric series to get a power series for $\frac{x^3}{6+x^2}$. Your answer needs to be a single sum of type $\sum c_n x^n$. State the interval of convergence (no need to check the endpoints).

$$\frac{x^3}{6+x^2} = x^3 \cdot \frac{1}{6\left(1+\frac{x^2}{6}\right)} = \frac{x^3}{6} \cdot \frac{1}{1-\left(-\frac{x^2}{6}\right)} = \frac{x^3}{6} \cdot \sum_{n=0}^{\infty} \left(-\frac{x^2}{6}\right)^n = \frac{x^3}{6} \sum_{n=0}^{\infty} \frac{(-1)^n}{6^n} x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} x^{2n+3}$$

converge for $\left|-\frac{x^3}{6}\right| < 1$

$|x|^3 < 6$

$|x| < \sqrt[3]{6}$

$-\sqrt[3]{6} < x < \sqrt[3]{6}$

6. (14pts) Recall that $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$.

a) Use a binomial series and antidifferentiation to find the McLaurin series for $\arcsin x$. Do not expand the binomial coefficient $\binom{k}{n}$ in the general sum.

b) Write the first four terms of the series in a), this time expanding the binomial coefficients and simplifying.

a) $(1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}(-x^2)^n = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}(-1)^n x^{2n}$

$\arcsin x = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1} + C$

When $x=0$ get $\arcsin 0 = 0 + C$ so $C=0$

b) $\arcsin x = x - \frac{(-\frac{1}{2})}{1}\frac{x^3}{3} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2}\frac{x^5}{5} - \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\frac{x^7}{7}$

$= x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots$

$\frac{15}{8}\cdot\frac{1}{6}\cdot\frac{1}{2} = \frac{5}{16}$

7. (16pts) Let $f(x) = \cos x$.

a) Find the 3rd Taylor polynomial for f centered at $a = \pi$.

b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Leave your answer as a fraction.

a)

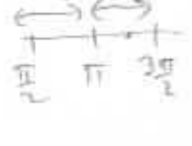
n	$f^{(n)}$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	

$T_3(x) = -1 + \frac{1}{2}(x-\pi)^2$

b) $R_3(x) = \frac{f^{(4)}(z)}{4!}(x-\pi)^4 = \frac{\cos z}{24}(x-\pi)^4$

$|R_3(x)| \le \frac{1}{24}|x-\pi|^4 \le \frac{1}{24}\left(\frac{\pi}{2}\right)^4 = \frac{\pi^4}{24\cdot 16} = \frac{\pi^4}{384}$

on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $|x-\pi| < \frac{\pi}{2}$



8. (16pts) Use the known McLaurin series for $\ln(1+x)$ to find the series representing $\int_0^1 \ln(1+x^2)\,dx$. Give an approximation of this definite integral with accuracy 10^{-2}. Write the approximation as a sum (you do not have to simplify it).

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$$

$$\int_0^1 \ln(1+x^2)\,dx = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{n}\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 x^{2n}\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{x^{2n+1}}{2n+1}\Big|_0^1$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(2n+1)} \leftarrow \text{decreasing terms, alt. series}$$

need $\frac{1}{n(2n+1)} < \frac{1}{100}$

$n(2n+1) > 100$

n	$n(2n+1)$
5	$5\cdot 11 = 55$
6	$6\cdot 13 = 78$
7	$7\cdot 15 = 105$ ✓

Sum up to $n=6$:

$$\frac{1}{1\cdot 3} - \frac{1}{2\cdot 5} + \frac{1}{3\cdot 7} - \frac{1}{4\cdot 9} + \frac{1}{5\cdot 11} - \frac{1}{6\cdot 13}$$

$$= \frac{1}{3} - \frac{1}{10} + \frac{1}{21} - \frac{1}{36} + \frac{1}{55} - \frac{1}{78}$$

Bonus (10pts) Let $a_n =$ the n-th digit of π, so $a_1 = 3$, $a_2 = 1$, $a_3 = 4$, etc. Find the interval of convergence for the series below. Don't forget to check the endpoints.

$$\sum_{n=1}^{\infty}\frac{(a_n+1)}{2^n}x^n$$

Root test $\sqrt[n]{\left|\frac{a_n+1}{2^n}x^n\right|} = \frac{\sqrt[n]{a_n+1}}{\sqrt[n]{2^n}}\sqrt[n]{|x|^n} = \frac{\sqrt[n]{a_n+1}}{2}|x| \to \frac{1}{2}|x|$

Converges if $\frac{1}{2}|x| < 1$, $|x| < 2$, $-2 < x < 2$

$1 \le a_n \le 10$

$\sqrt[n]{1} \le \sqrt[n]{a_n+1} \le \sqrt[n]{10}$

$\lim_{n\to\infty} \sqrt[n]{10} = 1$

$\lim_{n\to\infty} \sqrt[n]{1} = 1$

By squeeze thm

$\lim \sqrt[n]{a_n+1} = 1$

For $x=-2$ get $\sum_{n=1}^{\infty}\frac{a_n+1}{2^n}(-2)^n = \sum_{n=1}^{\infty}(-1)^n(a_n+1)$

$x = 2$ get $\sum_{n=1}^{\infty}\frac{a_n+1}{2^n}\cdot 2^n = \sum_{n=1}^{\infty}(a_n+1)$

In both cases $\lim_{n\to\infty}(a_n+1)$ does not exist

so series diverge by divergence test