Calculus 2 — Exam 3
MAT 308, Spring 2026 — D. Ivanšić

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Show all your work!

Find the limits, if they exist.

1. (6pts) $\lim_{n\to\infty} \frac{3^{2n+4}}{2^{3n-1}} = \lim_{n\to\infty} \frac{3^{2n}\cdot 3^4}{2^{3n}\,2^{-1}} = \lim_{n\to\infty} \frac{81\cdot(3^2)^n}{2^{-1}(2^3)^n} = \lim_{n\to\infty} 162\left(\frac{9}{8}\right)^n = 162\cdot\infty = \infty$

since $|r|>1$

$\lim r^n = \infty$

2. (6pts) $\lim_{n\to\infty} \sin\frac{(2n+1)\pi}{2} =$ does not exist

$\frac{(2n+1)\pi}{2} = (2n+1)\frac{\pi}{2}$ (odd multiples of $\frac{\pi}{2}$)

Sequence of $\{1,-1,1,-1,1,-1\dots\}$

3. (10pts) Find the limit. Use the theorem that rhymes with what you would unlock a door with.

$\lim_{n\to\infty} \frac{\cos(n^2-1)}{\sqrt{n}+\sqrt[3]{n}}$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt[3]{n}} = \frac{1}{\sqrt{\infty}+\sqrt[3]{\infty}} = \frac{1}{\infty+\infty} = 0$

$-1 \le \cos(n^2-1) \le 1$

$\lim_{n\to\infty} \frac{-1}{\sqrt{n}+\sqrt[3]{n}} = 0$ similarly

$\frac{-1}{\sqrt{n}+\sqrt[3]{n}} \le \frac{\cos(n^2-1)}{\sqrt{n}+\sqrt[3]{n}} \le \frac{1}{\sqrt{n}+\sqrt[3]{n}}$

By the squeeze theorem,

$\lim_{n\to\infty} \frac{\cos(n^2-1)}{\sqrt{n}+\sqrt[3]{n}} = 0$

4. (6pts) Write the series using sigma notation:

$$\frac{2}{1\cdot 2} \xrightarrow{+3} \frac{5}{1\cdot 2\cdot 3\cdot 4} \xrightarrow{+3} \frac{8}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{11}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{3n-1}{(2n)!}$$

even factorial,

5. (12pts) Justify why the series converges and find its sum.

$$\sum_{n=2}^{\infty}\frac{2^{2n-1}}{7^{n-2}} = \sum_{n=2}^{\infty}\frac{2^{2n}\cdot 2^{-1}}{7^n\cdot 7^{-2}} = \sum_{n=2}^{\infty}\frac{49\cdot 4^n}{2\cdot 7^n} = \sum_{n=2}^{\infty}\frac{49}{2}\left(\frac{4}{7}\right)^n$$

$\left|\frac{4}{7}\right| < 1$ so series converges

$$= \frac{\frac{49}{2}\left(\frac{4}{7}\right)^2}{1-\frac{4}{7}} = \frac{\frac{49}{2}\cdot\frac{16}{49}}{\frac{3}{7}} = 8\cdot\frac{7}{3} = \frac{56}{3}$$

Determine whether the following series converge and justify your answer.

6. (6pts) $\displaystyle\sum_{n=1}^{\infty}\frac{1-n^2}{n^2+5}$

$$\lim_{n\to\infty}\frac{1-n^2}{n^2+5} = \lim_{n\to\infty}\frac{\cancel{n^2}\left(\frac{1}{n^2}-1\right)}{\cancel{n^2}\left(1+\frac{5}{n^2}\right)} = \frac{0-1}{1+0} = -1 \leftarrow \text{not zero, so}$$

series diverges
by divergence test

7. (12pts) $\displaystyle\sum_{n=1}^{\infty}\frac{n^4+17}{n^6-n}$ like $\frac{n^4}{n^6}=\frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{\frac{n^4+17}{n^6-n}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\frac{\cancel{n^4}\left(1+\frac{17}{n^4}\right)}{n^2\cancel{n^4}\left(1-\frac{1}{n^5}\right)}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{1+\frac{17}{n^4}}{\cancel{n^2}\left(1-\frac{1}{n^5}\right)}\cdot\frac{\cancel{n^2}}{1} = \frac{1+0}{1-0} = 1$$

Since $\sum\frac{1}{n^2}$ converges, so does $\sum\frac{n^4+17}{n^6-n}$ by limit comparison test

$$\begin{array}{r} 86\cdot 86 \\ 516 \\ 688 \\ \hline 7396 \end{array} \qquad \begin{array}{r} 7396 \\ -\ 196 \\ \hline 7200 \end{array}$$

8. (20pts) Consider the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+7}$.

a) Is the series convergent? Justify.

b) Is the series absolutely convergent? Justify.

c) How many terms of the series do we need to add to find the sum with accuracy 0.1? (If you find the inequality too hard to solve, try some numbers that are easy to compute.)

a) Do alternating series test

$$f(x)=\frac{\sqrt{x}}{x+7} \quad f'(x)=\frac{\frac{1}{2\sqrt{x}}(x+7)-\sqrt{x}\cdot 1}{(x+7)^2}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}=\frac{x+7-2\sqrt{x}^2}{2\sqrt{x}(x+7)^2}=\frac{7-x}{2\sqrt{x}(x+7)^2}$$

$f'(x)\le 0$ For $x\ge 7$ so sequence is decreasing for $n\ge 7$

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+7}=\lim_{n\to\infty}\frac{1}{\sqrt{n}(1+\frac{7}{n})}=\frac{1}{\infty(1+0)}=\frac{1}{\infty}=0$$

By alt. series test series converges

b) $\sum\frac{\sqrt{n}}{n+7}$ is like $\frac{\sqrt{n}}{n}=\frac{1}{\sqrt{n}}$ $\quad\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n+7}}{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\frac{n}{n+7}=\lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}(1+\frac{7}{n})}=\frac{1}{1+0}=1$

Since $\sum\frac{1}{\sqrt{n}}$ diverges, by the limit comparison test $\sum\frac{\sqrt{n}}{n+7}$ diverges, too

c) $\frac{\sqrt{n}}{n+7}<\frac{1}{10}$

n	$\frac{\sqrt{n}}{n+7}$
49	$\frac{7}{56}>\frac{1}{10}$
81	$\frac{9}{88}>\frac{1}{10}$
100	$\frac{10}{107}<\frac{1}{10}$

Use 100 terms

Or: solve inequality:

$10\sqrt{n}<n+7$

$100n<n^2+14n+49$ $\quad 7200=2\cdot 3600$

$n^2-86n+49>0$

Sol: $\frac{-(-86)\pm\sqrt{(-86)^2-4\cdot 49}}{2}=\frac{86\pm 60\sqrt{2}}{2}$

$=43\pm 30\sqrt{2}$

≈ 88 for +

Determine whether the following series converge using the root or ratio test.

9. (11pts) $\sum_{n=1}^{\infty}\frac{2^{n+5}(\arctan n)^n}{3^{n+1}(n+7)}$ positive terms

$$\lim_{n\to\infty}\sqrt[n]{\frac{2^{n+5}(\arctan n)^n}{3^{n+1}(n+1)}}=\lim_{n\to\infty}\sqrt[n]{\frac{2^n\cdot 2^5(\arctan n)^n}{3^n\cdot(n+1)}}$$

$$=\lim_{n\to\infty}\frac{2\arctan n\cdot\sqrt[n]{32}}{3\sqrt[n]{n+1}}=\frac{2\frac{\pi}{2}\cdot 1}{3\cdot 1}=\frac{\pi}{3}>1$$

so series diverges by root test

10. (11pts) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n(4n+1)}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^n \frac{3^{n+1}(4n+5)}{(n+1)!}}{(-1)^{n-1}\frac{3^n(4n+1)}{n!}}\right| = \frac{\cancel{3^{n+1}}^{\,3}(4n+5)}{\cancel{(n+1)!}_{\,n+1}}\cdot\frac{\cancel{n!}}{\cancel{3^n}(4n+1)}$$

$$= \frac{3(4n+5)}{(n+1)(4n+1)} = \frac{3\cancel{n}(4+\frac{5}{n})}{n(1+\frac{1}{n})\cancel{n}(4+\frac{1}{n})} \rightarrow \frac{3(4+0)}{\infty(1+0)(4+0)} = \frac{3}{\infty} = 0 < 1$$

converges absolutely by the ratio test

Bonus. (10pts) Write the following infinite repeating decimal number as a fraction with integers in numerator and denominator.

$$0.24\,561\,561\,561\cdots = \frac{24}{100} + \frac{561}{100{,}000} + \frac{561}{100{,}000{,}000} + \cdots$$

$\cdot 1000 \qquad \cdot 1000$

$$= \frac{6}{25} + \sum_{n=1}^{\infty}\frac{561}{100}\left(\frac{1}{1000}\right)^n = \frac{6}{25} + \frac{\frac{561}{100}\cdot\frac{1}{1000}}{1-\frac{1}{1000}}$$

$$= \frac{6}{25} + \frac{561}{100{,}000}\cdot\frac{1000}{999} = \frac{6}{25} + \frac{\cancel{561}^{\,187}}{100\cdot\cancel{999}_{\,333}} = \frac{6\cdot 4\cdot 333 + 187}{100\cdot 333}$$

$$= \frac{7992+187}{100\cdot 333} = \frac{8179}{33300}$$

$$\begin{array}{r} 333\cdot 24 \\ \hline 666 \\ 1332 \\ \hline 7992 \end{array}$$