Calculus 2 — Exam 1
MAT 308, Spring 2026 — D. Ivanšić

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Show all your work!

Find the following integrals:

1. (12pts) 10 $\displaystyle\int x^2 e^x\,dx = \begin{bmatrix} u=x^2 & dv=e^x\,dx \\ du=2x\,dx & v=e^x \end{bmatrix} = x^2e^x - \int 2xe^x\,dx$

$$= \begin{bmatrix} u=x & dv=e^x\,dx \\ du=1 & v=e^x \end{bmatrix} = x^2e^x - 2\left(xe^x - \int e^x\,dx\right)$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

2. (8pts) $\displaystyle\int \sin^2 x\cos^5 x\,dx = \int \sin^2 x\,\underbrace{\cos^4 x}_{(1-\sin^2 x)^2}\,\cos x\,dx = \begin{bmatrix} u=\sin x \\ du=\cos x\,dx \end{bmatrix}$

$$= \int u^2(1-u^2)^2\,du = \int u^2(1-2u^2+u^4)\,du$$

$$= \int u^2-2u^4+u^6\,du = \frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3} = \frac{\sin^7 x}{7}-\frac{2\sin^5 x}{5}+\frac{\sin^3 x}{3}+C$$

Determine whether the following improper integral converges by calculating it directly.

3. (10pts) 14 $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\,dx = \begin{bmatrix} u=\ln x & dv=x^{-3}\,dx \\ du=\frac{1}{x}\,dx & v=\frac{x^{-2}}{-2} \end{bmatrix}$

$$= \lim_{t\to\infty}\left(-\frac{x^{-2}}{2}\ln x\Big|_1^t - \int_1^t \frac{x^{-2}}{-2}\cdot\frac{1}{x}\,dx\right)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}\left(\frac{\ln t}{t^2}-0\right)+\frac{1}{2}\int_1^t x^{-3}\,dx\right) = \lim_{t\to\infty}\left(-\frac{\ln t}{2t^2}+\frac{1}{2}\,\frac{x^{-2}}{-2}\Big|_1^t\right)$$

$$= \lim_{t\to\infty}\left(-\underbrace{\frac{\ln t}{2t^2}}_{\frac{\infty}{\infty}\leftarrow \text{L'H}} - \frac{1}{4}\left(\frac{1}{t^2}-1\right)\right) = \lim_{t\to\infty} -\frac{\frac{1}{t}}{4t} - \frac{1}{4\cdot\infty}+\frac{1}{4} = \underbrace{-\frac{1}{\infty}}_{=0}+\frac{1}{4} = \frac{1}{4}$$

$$= -\frac{1}{4t^2}+0+4=$$

so integral converges

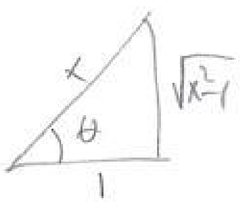
Use trigonometric substitution to evaluate the following integrals. Don't forget to return to the original variable where appropriate.

4. (~~14~~pts) 12 $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \left[\begin{array}{l} x = \sec\theta \\ dx = \sec\theta\tan\theta \end{array}\right] = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\underbrace{\sqrt{\sec^2\theta-1}}_{\tan^2\theta}}$

$$= \int \frac{\cancel{\sec\theta}\ \cancel{\tan\theta}}{\sec^2\theta\ \cancel{\tan\theta}}\,d\theta = \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta$$

$$= \frac{\sqrt{x^2-1}}{x} + C$$

$=$



$\sec\theta = x$

$\cos\theta = \frac{1}{x}$

$\sin\theta = \frac{\sqrt{x^2-1}}{x}$

5. (14pts) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \left[\begin{array}{lll} x = 2\sin\theta & 1 = 2\sin\theta \quad \frac{1}{2} = \sin\theta, & \theta = \frac{\pi}{6} \\ dx = 2\cos\theta\,d\theta & 0 = 2\sin\theta & \theta = 0 \end{array}\right]$

$$= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\underbrace{\sqrt{4-4\sin^2\theta}}_{4\cos^2\theta}} \cdot 2\cos\theta\,d\theta = \int_0^{\frac{\pi}{6}} \frac{8\sin^2\theta\,\cancel{\cos\theta}}{2\cancel{\cos\theta}}\,d\theta = 4\int_0^{\frac{\pi}{6}} \sin^2\theta\,d\theta$$

$$= 4\int_0^{\frac{\pi}{6}} \frac{1-\cos(2\theta)}{2}\,d\theta = 2\left(\left(\frac{\pi}{6}-0\right) - \frac{\sin(2\theta)}{2}\Bigg|_0^{\frac{\pi}{6}}\right)$$

$$= 2\left(\frac{\pi}{6} - \frac{1}{2}\left(\underbrace{\frac{\sqrt{3}}{2}}_{\sin\frac{\pi}{3}} - 0\right)\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

Use the method of partial fractions to find the integral.

6. (14pts) $\int \frac{6x^2+x+33}{(2x+3)(x^2+9)}\,dx =$

$$\frac{6x^2+x+33}{(2x+3)(x^2+9)} = \frac{A}{2x+3} + \frac{Bx+C}{x^2+9}$$

$$6x^2+x+33 = A(x^2+9)+(2x+3)(Bx+C)$$

$$33 = 9A+3C \Rightarrow 11 = 3A+C, \quad C = 11-3A$$

$$1 = 3B+2C \qquad 1 = 3B+2(11-3A)$$

$$6 = A+2B \qquad 1 = 3B+22-6A$$

$$\Rightarrow A = 6-2B \qquad 1 = 3B+22-6(6-2B)$$

$$-21 = 3B-36+12B$$

$$15 = 15B \quad B=1, \quad A = 6-2=4, \quad C = 11-3\cdot 4 = -1$$

$$= \int \frac{4}{2x+3} + \frac{x-1}{x^2+9}\,dx = 2\ln|2x-3| + \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\arctan\frac{x}{3} + C$$

$4 \cdot \frac{1}{2}$

7. (10pts) Use comparison to determine whether the improper integral $\int_1^\infty \frac{x^2}{x^4+6}\,dx$ converges.

$$\frac{x^2}{x^4+6} \le \frac{x^2}{x^4} = \frac{1}{x^2}$$

smaller denom

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, by comparison test,

so does $\int_1^\infty \frac{x^2}{x^4+6}\,dx$

8. (18pts) Suppose we wanted to approximate the number $\ln 2$. We could do it by approximating the integral $\int_{\frac{1}{2}}^{1} \frac{1}{x}\,dx = \ln 2$, which uses only the four algebraic operations.
a) Write the expression you would use to calculate M_4, the midpoint rule with 4 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) Find the error estimate for M_n in general. You will need the second derivative of $\frac{1}{x}$.
c) Estimate the error for M_4.
d) What should n be in order for M_n to give you an error less than 10^{-4}?

a) $\frac{9}{16}$ $\frac{11}{16}$ $\frac{13}{16}$ $\frac{15}{16}$

$\frac{1}{2}$ $\frac{5}{8}$ $\frac{3}{4}$ $\frac{7}{8}$ 1

$$M_4 = \frac{1}{8}\left(\frac{1}{\frac{9}{16}} + \frac{1}{\frac{11}{16}} + \frac{1}{\frac{13}{16}} + \frac{1}{\frac{15}{16}}\right)$$

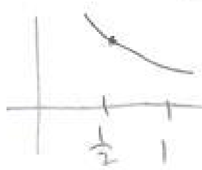
$$\Delta x = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}$$

$$|E_M| \le \frac{16\cdot\left(1-\frac{1}{2}\right)^3}{24\cdot n^2} = \frac{16\cdot\frac{1}{8}}{24n^2} = \frac{2}{24n^2} = \frac{1}{12n^2}$$

b) $y = x^{-1}$

$y' = -x^{-2}$

$y'' = 2x^{-3} = \frac{2}{x^3}$



has max at $x = \frac{1}{2}$

$$K_2 = \frac{2}{\left(\frac{1}{2}\right)^3} = \frac{2}{\frac{1}{8}} = 16$$

c) For $n=4$, $|E_M| \le \frac{1}{12\cdot 16} = \frac{1}{192}$

d) Need $\frac{1}{12n^2} < 10^{-4}$

$\frac{10^4}{12} < n^2$

$n \ge \sqrt{\frac{10^4}{12}} = \frac{100}{\sqrt{12}}$

Bonus (10pts) Show the reduction formula.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \sin^n x\,dx = \int \sin^{n-1} x \cdot \sin x\,dx = \left[\begin{matrix} u = \sin^{n-1} x & dv = \sin x \\ du = (n-1)\sin^{n-2} x\cos x\,dx & v = -\cos x \end{matrix}\right]$$

$$= -\sin^{n-1} x\cos x - \int (n-1)\sin^{n-2} x\cos x\,(-\cos x)\,dx$$

$$= -\sin^{n-1} x\cos x + \int (n-1)\sin^{n-2} x\underbrace{\cos^2 x}_{1-\sin^2 x}\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\underbrace{\int \sin^n x\,dx}_{I}$$

$$I = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)I \quad | + (n-1)I$$

$$nI = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx \quad | \div n$$

$$I = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$