Real Function Theory I — Exam 1 **MAT 726, Spring 2025 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) State theorem on continuity of measure for ascending sets.

Theory 2. (3pts) State the Simple Approximation Lemma (not the theorem).

Theory 3. (3pts) State Egoroff's Theorem on pointwise and uniform convergence.

Type A problems (5pts each)

A1. Give an example of a descending sequence of measurable sets $(B_n, n \in \mathbf{N})$ where $mB_1 = \infty$ and the conclusion of continuity of measure fails.

A2. Show that every measurable set with positive measure is a disjoint union of two non-measurable sets.

A3. Let E be measurable and let $f : E \to \mathbf{R}$ be a measurable function. Explain why the function $g : E \to \mathbf{R}$, $g(x) = 3f(x) - (\max\{f(x), \sin x\})^2$ is measurable.

A4. Let $f : E \to \mathbf{R}$ be bounded and E measurable. Use the Simple Approximation Lemma to show there exists a sequence of functions $f_n : E \to \mathbf{R}$ such that $f_n \to f$ uniformly on E.

A5. Recall Thomae's function $f : (0,1) \to \mathbf{R}$, $f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n}, \text{ reduced} \\ 0, & \text{if } x \notin \mathbf{Q} \end{cases}$. Use the definition to show this function is measurable.

Type B problems (8pts each)

B1. Let E be measurable. Show a function $f : E \to \mathbf{R}$ is measurable if and only if $f^{-1}(A)$ is measurable for every Borel set A. *(Hint: show the collection of sets A for which $f^{-1}(A)$ is measurable is a σ-algebra.)*

B2. Give an example of a function $f : [0,1] \to \mathbf{R}$ such that for every $c \in \mathbf{R}$, $f^{-1}(c)$ is measurable, but the function f is not measurable. *(Hint: any injective function satisfies the assumption.)*

B3. Given Thomae's function $f : (0,1) \to \mathbf{R}$ (see **A5**), construct a sequence of simple functions f_n that converges to f pointwise. (Such a sequence exists since f is measurable, according to **A5**).

B4. Let $f_n : [0,1] \to \mathbf{R}$, $f_n(x) = x^n$. Explain why $f_n \to f$ pointwise on $[0,1]$, but not uniformly (what is f?). Given ε, determine the closed set F from Egoroff's theorem on which $f_n \to f$ uniformly on F, where $m([0,1] - F) < \varepsilon$. Good pictures with explanations will suffice.

B5. Let $f : [0,1] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} (-1)^n, & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \text{ for every } n \in \mathbf{N} \\ 0, & \text{if } x = 0 \end{cases}$$

Show that f is a measurable function, and, given ε, determine the closed set F whose existence is guaranteed by Lusin's theorem, such that $m([0,1]-F) < \varepsilon$ and $f|_F$ is continuous.

B6. Let $\psi(x) = x+\phi(x)$, where ϕ is the Cantor-Lebesgue function. Recall that $\psi : [0,1] \to [0,2]$ sends both the Cantor set and its complement $U = [0,1] - C$ to sets of measure 1, consequently sending a subset of a Cantor set to a nonmeasurable set. Modify this example to show that ψ sends a measurable set of positive measure to a nonmeasurable set.

Real Function Theory I — Exam 2 **MAT 726, Spring 2025 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Let $f : E \to \overline{\mathbf{R}}$. Assuming integrals of positive functions have been defined, define the integral of a general function, if it exists.

Theory 2. (3pts) State the Monotone Convergence Theorem.

Theory 3. (3pts) State the Vitali Convergence Theorem.

Type A problems (5pts each)

A1. Determine if the function $f : (0,1] \to \mathbf{R}$, $f(x) = \frac{1}{x^2}$ is integrable over $(0,1]$. If it is, determine $\int_{(0,1]} f$. Justify your work with theory.

A2. Use the Darboux sum definition of the Riemann integral to give an example of a bounded function $f : [0,1] \to \mathbf{R}$ that is not Riemann integrable.

A3. Give an example of a sequence of functions $f_n : E \to \mathbf{R}$, $f_n \geq 0$, such that $f_n \to f$ pointwise on E, and $\int_E f < \liminf \int_E f_n$ (example of a strict inequality in Fatou's lemma).

A4. Suppose $f_n : E \to \overline{\mathbf{R}}$, $f_n \geq 0$, is a sequence of functions that converges pointwise to $f : E \to \overline{\mathbf{R}}$. Use Fatou's lemma to show: If $f_n \leq f$ on E for all n, show that $\displaystyle\lim_{n\to\infty} \int_E f_n = \int_E f$.

A5. Show how countable additivity of integration implies continuity of integration.

A6. Let $\mathcal{F}$ and $\mathcal{G}$ be two uniformly integrable families of functions $f : E \to \overline{\mathbf{R}}$. Let $\alpha, \beta \in \mathbf{R}$ and define $\mathcal{H} = \{\alpha f + \beta g \mid f \in \mathcal{F}, g \in \mathcal{G}\}$. Show that the family of functions $\mathcal{H}$ is uniformly integrable.

Type B problems (8pts each)

B1. Show that the uniform convergence theorem fails if $mE = \infty$. In other words, give an example of a sequence $f_n : E \to \mathbf{R}$, $mE = \infty$, of integrable functions that conveges uniformly to a function $f : E \to \mathbf{R}$, but $\displaystyle\lim_{n\to\infty} \int_E f_n \neq \int_E f$.

B2. Use the Darboux sum definition of the Riemann integral to show that the function $f(x) = \begin{cases} 1, & \text{if } x \in [0,1) \\ 5, & \text{if } x \in [1,3] \end{cases}$ is integrable on $[0,3]$ and determine the integral.

B3. Let $f : [1,\infty) \to \mathbf{R}$, $f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \in [n, n+1),\ n \text{ odd} \\ -\frac{2}{x^2}, & \text{if } x \in [n, n+1),\ n \text{ even} \end{cases}$ Determine whether f is integrable and, if it is, find its integral (expressed as a sum of a series). Justify your work with theory.

B4. If $h : E \to \mathbf{R}$, $mE < \infty$, is a measurable and bounded function, the Simple Approximation Lemma tells us that for every $\varepsilon > 0$ there exist simple functions $\phi \leq h \leq \psi$ on E such that $\psi - \phi < \varepsilon$. Show that ϕ and ψ can be chosen so that the sets $E_1, \ldots, E_n$ and $F_1, \ldots, F_m$ appearing in the canonical representations of ϕ and ψ are the same. (Recall the canonical representation of a simple function: $\phi = \sum_{i=1}^{n} c_i \chi_{E_i}$ where $c_1, \ldots, c_n$ are distinct and $E_1, \ldots, E_n$ are pairwise disjoint).

B5. Let $f_n : E \to \overline{\mathbf{R}}$, $n \in \mathbf{N}$, be a sequence of functions that converges uniformly to $f : E \to \overline{\mathbf{R}}$. If f is integrable, show that $\lim_{n\to\infty} \int_E f_n = \int_E f$. (Hint: Lebesgue Dominated Convergence Theorem.)

B6. Let $f_n : E \to \overline{\mathbf{R}}$, $f_n \geq 0$, $n \in \mathbf{N}$, be a decreasing sequence of functions where f_1 is integrable. Show that $\limsup \int_E f_n \leq \int_E \limsup f_n$.

Type C problems (12pts each)

C1. Let $\mathcal{F}$ be a uniformly integrable family of functions $f : E \to \overline{\mathbf{R}}$, and let $g : E \to \overline{\mathbf{R}}$ be a measurable function, and let $\mathcal{H} = \{fg \mid f \in \mathcal{F}\}$.
a) If g is bounded, show that $\mathcal{H}$ is uniformly integrable.
b) If g is integrable over E, does it follow that $\mathcal{H}$ is uniformly integrable?