Real Function Theory I — Exam 3 MAT 726, Spring 2016 — D. Ivanšić	Name: ________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) If E is measurable, define the set L^pE.

Theory 2. (3pts) State Holder's inequality, along with the the statement about f^*.

Theory 3. (3pts) Define a Cauchy sequence in a normed linear space.

Type A problems (5pts each)

A1. On $C[a,b]$ define $||f||_1 = \int_{[a,b]} |f|$. Show that $||\ ||_1$ is a norm.

A2. For every $p \in [1,\infty]$, give an example of a function $f \in L^p[2,\infty)$ such that $f(x) > 0$ for all $x \in [2,\infty)$.

A3. Give an example of a function that is in $L^3(0,1)$, but is not in $L^5(0,1)$.

A4. If $f, g \in L^pE$, does it follow that the product fg is in L^pE?

A5. Let E be measurable, $mE < \infty$ and $S \subset L^1E$ be the subspace of simple functions with finite support. Is S a Banach space with respect to $||\ ||_1$?

A6. Let $f \in L^{p_1}E$, and let f be bounded. If $p_2 > p_1$, show that $f \in L^{p_2}E$. (This statement is different from the similar statement we had that assumed $mE < \infty$, but did not assume that f was bounded. Here, mE may be infinite. Don't do anything hard: this one needs only a little algebra.)

Type B problems (8pts each)

B1. Prove Holder's inequality for three functions: let $p, q, r \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$. If $f \in L^pE$, $g \in L^qE$ and $h \in L^rE$, then $\int_E |fgh| \leq ||f||_p||g||_q||h||_r$. (Start by using Holder's inequality on functions fg and h. First show that $fg \in L^{p'}E$, where $\frac{1}{p'} = \frac{1}{p} + \frac{1}{q}$.)

B2. Let $\mathcal{P}_n$ be the linear space of polynomials of degree $\leq n$, and $b_0, \dots, b_n$ a collection of $n+1$ distinct real numbers. Show that the functional $||\ || : \mathcal{P}_n \to \mathbf{R}$ is a norm on $\mathcal{P}_n$, where $||f|| = |f(b_0)| + |f(b_1)| + \cdots + |f(b_n)|$.

B3. Give an example of a convergent sequence $\{a_n \mid n \in \mathbf{N}\}$ of real numbers so that there does not exist a convergent series $\sum \epsilon_n$ satisfying $|a_k - a_{k+1}| \leq \epsilon_k$ for every $k \in \mathbf{N}$. (Your sequence cannot be monotone, since in this case convergence of $\{a_n\}$ is equivalent to convergence of the series $\sum |a_k - a_{k+1}|$, in which case you could use $\epsilon_k = |a_k - a_{k+1}|$.)

B4. Let $mE < \infty$ and $1 \leq p_1 < p_2 \leq \infty$. Corollary 7.3 states that for an $f \in L^{p_2}E$, there is a constant $c > 0$ such that $||f||_{p_1} \leq c||f||_{p_2}$. Show that there is no constant c satisfying $||f||_{p_2} \leq c||f||_{p_1}$ for all $f \in L^{p_2}E$, by examining the family of functions $\left\{ \frac{1}{x^{\frac{\alpha}{p_2}}} \mid 0 < \alpha < 1 \right\}$, on $E = (0,1]$.

B5. Let $f_n \to f$ pointwise on E, where $0 \leq f_n \leq f$ on E and $f \in L^p E$. Show that $f_n \in L^p E$ for every $n \in \mathbf{N}$, and that $f_n \to f$ in $L^p E$.

B6. Show that every Cauchy sequence has a rapidly Cauchy subsequence.

Type C problems (12pts each)

C1. Let $C \subset l^\infty$ be the subspace of all sequences that converge to a real number. Show that this is a Banach space with the norm $|| \; ||_\infty$.