Real Function Theory I — Exam 2
MAT 726, Spring 2016 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Let $f : E \to \overline{\mathbf{R}}$, $f \geq 0$. Define the integral of a nonnegative function.

Theory 2. (3pts) State Fatou's Lemma.

Theory 3. (3pts) State the Lebesgue Dominated Convergence Theorem.

Type A problems (5pts each)

A1. Determine if the function $f : [1,\infty) \to \mathbf{R}$, $f(x) = \frac{1}{x^3}$ is integrable over $[1,\infty)$. If it is, determine $\int_{[1,\infty)} f$. Justify your work with theory.

A2. Give an example of a sequence of functions $f_n : E \to \mathbf{R}$ such that $f_n \to f$ pointwise on E, but $\int_E f_n$ does not converge to $\int_E f$.

A3. Give a counterexample to the Bounded Convergence Theorem if we remove the assumption that $mE < \infty$, that is, give an example of a sequence of functions $f_n : E \to \mathbf{R}$, each with finite support, such that $|f_n| < M$ for all $n \in \mathbf{N}$, but $\int_E f_n$ does not converge to $\int_E f$.

A4. Suppose $f_n : E \to \overline{\mathbf{R}}$, $n \in \mathbf{N}$, $f : E \to \overline{\mathbf{R}}$ are all integrable over E and that $f_n \to f$ pointwise on E. Show that if $\int_E |f_n - f| \to 0$, then $\int_E f_n \to \int_E f$.

A5. Show that countable additivity of integration holds if $f \geq 0$, without the assumption that f is integrable.

A6. Let $\mathcal{F}$ and $\mathcal{G}$ be two uniformly integrable families of functions $f : E \to \overline{\mathbf{R}}$. Let $\alpha, \beta \in \mathbf{R}$ and define $\mathcal{H} = \{\alpha f + \beta g \mid f \in \mathcal{F}, g \in \mathcal{G}\}$. Show that the family of functions $\mathcal{H}$ is uniformly integrable.

Type B problems (8pts each)

B1. Let $f : [1,\infty) \to \mathbf{R}$, $f(x) = \frac{1}{n^2}$ for $x \in [n, n+\frac{1}{2})$, and $f(x) = -\frac{1}{3n^2}$, for $x \in [n+\frac{1}{2}, n+1)$, for every $n \in \mathbf{N}$. Determine whether f is integrable and, if it is, find its integral (expressed as a sum of a series). Justify your work with theory.

B2. Let $f : E \to \overline{\mathbf{R}}$, $f \geq 0$ be integrable. Given $\epsilon > 0$, show there exists a simple function ϕ of finite support, $0 \leq \phi \leq f$, such that $\int_E f - \int_E \phi < \epsilon$.

B3. Suppose $f_n : E \to \overline{\mathbf{R}}$, $f_n \geq 0$ on E, $n \in \mathbf{N}$, is a sequence of functions. Show the generalized Fatou Lemma: $\int_E \liminf f_n \leq \liminf \int_E f_n$.

B4. Suppose $f_n : E \to \overline{\mathbf{R}}$, $f_n \geq 0$ on E, $n \in \mathbf{N}$, is a *decreasing* sequence of functions that converges pointwise to $f : E \to \overline{\mathbf{R}}$. If f_1 is integrable, show that $\int_E f_n \to \int_E f$. Give an example where the conclusion does not hold if f_1 is not integrable.

B5. Give an example where countable additivity of integration fails, if f is not assumed to be integrable.

B6. Let $f : E \to \overline{\mathbf{R}}$ be integrable. Show that f^2 is integrable, where $f^2(x) = (f(x))^2$. (Hint: start with $\{x \in E \mid |f(x)| > 1\}$, apply Chebyshev's inequality, and then additivity over domains.)

Type C problems (12pts each)

C1. Let $\mathcal{F}$ be a uniformly integrable family of functions $f : E \to \overline{\mathbf{R}}$, and let $g : E \to \overline{\mathbf{R}}$ be a measurable function, and let $\mathcal{H} = \{fg \mid f \in \mathcal{F}\}$.
a) If g is bounded, show that $\mathcal{H}$ is uniformly integrable.
b) If g is integrable over E, does it follow that $\mathcal{H}$ is uniformly integrable?

C2. Let a bounded function $f : E \to \mathbf{R}$, $mE < \infty$, be integrable. The definition of integrability in this case does not assume that f is measurable. Show that f is measurable by using these steps:
a) Show there exist simple functions $\phi_n, \psi_n : E \to \mathbf{R}$ such that $\phi_n \leq f \leq \psi_n$ on E and $\int_E(\psi_n - \phi_n) < \frac{1}{2^{2n}}$, for every $n \in \mathbf{N}$.
b) Define $E_{mn} = \{x \in E \mid \psi_n(x) - \phi_n(x) > \frac{m}{2^n}\}$ and $F_m = \cup_{n\in\mathbf{N}} E_{mn}$. Use Chebyshev's inequality and countable additivity to show that $m(E_{mn}) < \frac{1}{2^n m}$ and $m(F_m) < \frac{1}{m}$.
c) Show that if $x \in F_m^c$, then $\psi_n(x) - \phi_n(x) \to 0$, so $\psi_n(x), \phi_n(x) \to f(x)$.
d) Observe that F_m is a descending sequence and show that $\psi_n - \phi_n \to 0$ ae on E.
e) Conclude that f is measurable function as the limit ae of simple, hence measurable, functions.