Real Function Theory I — Exam 1 MAT 726, Spring 2016 — D. Ivanšić	Name: ____________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) State one of the four equivalent definitions of a measurable function.

Theory 2. (3pts) State the Simple Approximation Theorem.

Theory 3. (3pts) Define a lower Darboux sum.

Type A problems (5pts each)

A1. Let $f : E \to \mathbf{R}$ be a defined on a measurable set E so that $E = F \cup G$, where F and G are disjoint, $mG = 0$ and $f|_F : F \to \mathbf{R}$ is continuous. Show that $f : E \to \mathbf{R}$ is measurable.

A2. Given the function $f : (0, 1] \to \mathbf{R}$, $f(x) = \frac{1}{x}$, construct a sequence of step-functions that converges to f pointwise. A good picture with an explanation will suffice. (The existence of such a sequence is warranted by the Simple Approximation Theorem).

A3. Let $f : E \to \mathbf{R}$ be bounded and E measurable. Use the Simple Approximation Lemma to show there exists a sequence of functions $f_n : E \to \mathbf{R}$ such that $f_n \to f$ uniformly on E.

A4. Let $f : E \to \mathbf{R}$ be a simple function, $g : \mathbf{R} \to \mathbf{R}$ any function. Show that $g \circ f$ is a simple function. (Don't forget the part about measurability.)

A5. Let $f : [a, b] \to \mathbf{R}$ be bounded. Is there an upper bound for all upper Darboux sums $U(f, \mathcal{P})$, or a lower bound for all lower Darboux sums $L(f, \mathcal{P})$? Justify.

Type B problems (8pts each)

B1. Show that the Cantor set C has the property: for every $x, y \in C$, $x < y$, there exists a $t \notin C$ such that $x < t < y$. (Because of this, we say that C is *totally disconnected*.)

B2. Let $f : E \to \mathbf{R}$, where E is measurable, be a function such that $f^{-1}([a, b])$ is a measurable set for every $a, b \in \mathbf{R}$, $a < b$. Show that f is a measurable function.

B3. Let $f_n : E \to \mathbf{R}$ be a sequence of measurable functions. Show that the function $\sup f_n$ is measurable.

B4. Let $f_n : [0, 1] \to \mathbf{R}$ be defined by $f_n(x) = \begin{cases} nx, & \text{if } x \in [0, \frac{1}{n}] \\ 1, & \text{if } x \in (\frac{1}{n}, 1] \end{cases}$. Explain why $f_n \to f$ pointwise on $[0, 1]$, but not uniformly (what is f?). Given ϵ, determine the closed set F from Egoroff's theorem on which $f_n \to f$ uniformly on F, where $m([0, 1] - F) < \epsilon$. Good pictures with explanations will suffice.

B5. Let C be the Cantor set, and let $f : [0, 1] \to \mathbf{R}$ be defined by $f(x) = \begin{cases} x, & \text{if } x \notin C \\ 0, & \text{if } x \in C \end{cases}$. Show that f is a measurable function, and, given ϵ, determine the closed set F whose existence is guaranteed by Lusin's theorem, such that $m([0, 1] - F) < \epsilon$ and $f|_F$ is continuous.

B6. Prove that a bounded function $f : [a, b] \to \mathbf{R}$ is Riemann-integrable if and only if for every $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.

Type C problems (12pts each)

C1. Let $\{q_n \mid n \in \mathbf{N}\}$ be an enumeration of rational numbers in $[0,1]$ and let a_n be a sequence whose limit is L and for which $|a_n| \leq M$, for all $n \in \mathbf{N}$. Show that the function $f : [0,1] \to \mathbf{R}$ defined by $f(x) = \begin{cases} a_n, & \text{if } x = q_n \\ L, & \text{if } x \notin \mathbf{Q} \cap [0,1] \end{cases}$ is Riemann-integrable by following the steps:

a) Given $\epsilon > 0$, show there exists an $n_0 \in \mathbf{N}$ such that $|a_n - L| < \epsilon$ and $\frac{4M}{n} < \epsilon$ for all $n \geq n_0$.

b) For an $n \geq n_0$, consider the partitition $\mathcal{P}$ of $[0,1]$ consisting of n^2 equal-width subintervals. Show that in at most $2n$ of those subintervals we have $M_i - m_i \leq 2M$, and that $M_i - m_i \leq 2\epsilon$ holds for the rest of the subintervals. Use this to show that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < 3\epsilon$.

c) Conclude that f is Riemann-integrable.

(Note: Thomae's function is a special case of this one.)