4.1 The Riemann Integral

Recall the definition of the Riemann integral $\int_a^b f(x)dx$.

Definition. Let $a = x_0 < x_1 < \cdots < x_n = b$, $t_i \in [x_{-1}, x_i]$ be a tagged partition of $[a,b]$. $f : [a,b] \to \mathbf{R}$ is Riemann integrable if there exists an a number $L \in \mathbf{R}$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ s.t. if $\dot{\mathcal{P}}$ is any tagged partition with $||\dot{\mathcal{P}}|| < \delta$, then $|S(f, \dot{\mathcal{P}}) - L| < \epsilon$, where $S(f, \dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)\Delta x_i$. and $\Delta x_i = x_i - x_{i-1}$.

Here is an alternative approach:

Definition. Let $f : [a,b] \to \mathbf{R}$ be bounded, $\mathcal{P}$ a partition of $[a,b]$. We define the

lower Darboux sum: $$L(f, \mathcal{P}) = \sum_{i=1}^n m_i \Delta x_i, \text{ where } m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

upper Darboux sum: $$U(f, \mathcal{P}) = \sum_{i=1}^n M_i \Delta x_i, \text{ where } M_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]$$

and the

lower Riemann integral: $$\int_{\underline{a}}^{b} f = \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ is a partition of } [a,b]\}$$

upper Riemann integral: $$\int_{a}^{\bar{b}} f = \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ is a partition of } [a,b]\}$$

A function is Riemann integrable if $\displaystyle\int_{\underline{a}}^{b} f = \int_{a}^{\bar{b}} f$.

Theorem 4.0. A function is Riemann-integrable via Riemann sum definition if and only if it is Riemann-integrable via the Darboux sum definition.

Proof.

Example. Recall the piecewise linear functions $f_n : [0,1] \to \mathbf{R}$ passing through points $(0,0)$, $\left(\frac{1}{n}, n\right)$, $\left(\frac{2}{n}, 0\right)$ and $(1,0)$.

$f_n \to 0$, yet $\displaystyle\int_0^1 f_n = 1 \nrightarrow 0$

Example. Let $f : [0,1] \to \mathbf{R}$ be the Dirichlet function, (q_n) a sequence enumerating rational numbers in $[0,1]$, and let $f_n : [0,1] \to \mathbf{R}$ functions given by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q} \\ 0, & \text{if } x \notin \mathbf{Q} \end{cases} \qquad f_n(x) = \begin{cases} 1, & \text{if } x \in \{q_1, \dots q_n\} \\ 0, & \text{if } x \notin \{q_1, \dots q_n\} \end{cases}$$

Show that f is not Riemann integrable via the Darboux sums and that $f_n \to f$, giving a sequence of Riemann-integrable functions that converges to a function that is not Riemann-integrable.

4.2 The Lebesgue Integral of a Measurable Function...

Definition. Let E be measurable and let $\psi : E \to \mathbf{R}$ be a simple function that takes on takes on distinct values $a_1, \ldots, a_n \in \mathbf{R}$. Then $\psi = \sum_{i=1}^{n} a_i \chi_{E_i}$ where $E_i = f^{-1}(a_i)$ are disjoint. This is called the *canonical representation of* ψ.

Example. Show how $\psi : [1,3] \to \mathbf{R}$, $\psi = 4\chi_{[1,2)} + 5\chi_{[2,3]}$ can be written in other ways $\sum_{i=1}^{n} a_k \chi_{E_i}$ where E_k's are not disjoint.

Because the canonical representation of a simple function is unique, we may define:

Definition. For a simple functon $\psi : E \to \mathbf{R}$, where $mE < \infty$, we define the integral of ψ over E by

$$\int_E \psi = \sum_{i=1}^{n} a_i \cdot mE_i, \text{ where } \psi = \sum_{i=1}^{n} a_i \chi_{E_i} \text{ is the canonical representation of } \psi$$

Lemma 4.1. Let $\psi = \sum_{i=1}^{n} a_i \chi_{E_i}$, where the E_i's are disjoint and measurable, $E_i \subseteq E$, $mE < \infty$. Then $\int_E \psi = \sum_{i=1}^{n} a_i \cdot mE_i$

Proof. The difference with the original formula is that not all a_i need be distinct — see book.

Proposition 4.2. Let $\varphi, \psi : E \to \mathbf{R}$ be simple functions, $mE < \infty$. Then for any $\alpha, \beta \in \mathbf{R}$ we have:

$$\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi \qquad \text{if } \varphi \le \psi \text{ on } E, \text{ then } \int_E \varphi \le \int_E \psi$$

Proof.

Example. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $E = [a, b]$, and let $f : E \to \mathbf{R}$ be the step function $f(x) = \begin{cases} a_i, & \text{if } x \in (x_{i-1}, x_i) \\ b_i, & \text{if } x = x_i \end{cases}$. Show that the Lebesgue integral of this simple function is equal to its Riemann integral.

To define the Lebesgue integral, we follow the pattern for defining the Riemann integral via Darboux sums.

Definition. Let $f : E \to \mathbf{R}$ be bounded, $mE < \infty$. We define the

$$\textit{lower Lebesgue integral:} \quad \int_{\underline{E}} f = \sup \left\{ \int_E \varphi \mid \varphi \text{ simple and } \varphi \leq f \text{ on } E \right\}$$

$$\textit{upper Lebesgue integral:} \quad \int_{\overline{E}} f = \inf \left\{ \int_E \psi \mid \psi \text{ simple and } f \leq \psi \text{ on } E \right\}$$

Note that if $m \le f \le M$ on E, then $m \cdot mE \le \underline{\int_E} f \le \overline{\int_E} f \le M \cdot mE$, since $\int_E \varphi \le \int_E \psi$, owing to $\varphi \le f \le \psi$.

Definition. We say that a bounded function $f : E \to \mathbf{R}$, $mE < \infty$, is *Lebesgue-integrable over* E if $\underline{\int_E} f = \overline{\int_E} f$. In this case we set $\int_E f = \underline{\int_E} f = \overline{\int_E} f$

Theorem 4.3. If $f : [a, b] \to \mathbf{R}$ is Riemann-integrable, then it is Lebesgue-integrable.

Proof.

Note. $\chi_{\mathbf{Q}} : [0, 1] \to \mathbf{R}$ is Lebesgue-integrable over $[0, 1]$, but is not Riemann-integrable over $[0, 1]$, and $\int_{[0,1]} \chi_{\mathbf{Q}} = 1 \cdot m(\mathbf{Q} \cap [0, 1]) = 0$.

Theorem 4.4. If $mE < \infty$ and $f : E \to \mathbf{R}$ is bounded and measurable, then f is Lebesgue-integrable.

Proof.

Theorem 4.5. If $mE < \infty$ and $f, g : E \to \mathbf{R}$ be bounded and measurable. Then

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g \qquad \text{if } f \leq g \text{ on } E, \text{ then } \int_E f \leq \int_E g$$

Proof.

Corollary 4.6. Let $mE < \infty$ and let $f : E \to \mathbf{R}$ be bounded and measurable, $A, B \subseteq E$, disjoint and measurable. Then $\displaystyle\int_{A \cup B} f = \int_A f + \int_B f$.

Proof.

Corollary 4.7. Let $mE < \infty$ and let $f : E \to \mathbf{R}$ be bounded and measurable. Then

$$\left| \int_E f \right| \leq \int_E |f|$$

Proof.

Proposition 4.8. Let $mE < \infty$ and let $(f_n) : E \to \mathbf{R}$ be a sequence of bounded and measurable functions. If $f_n \to f$ uniformly on E, then $\int_E f_n \to \int_E f$.

Proof.

Note. Pointwise convergence is not enough to claim $\int_E f_n \to \int_E f$, as in example in 4.1.

Bounded Convergence Theorem. Let $mE < \infty$ and let $(f_n) : E \to \mathbf{R}$ be a sequence of measurable functions that is uniformly pointwise bounded, meaning there exists an $M \in \mathbf{R}$ such that $|f_n| \leq M$ for all $n \in \mathbf{N}$. If $f_n \to f$ pointwise on E, then $\int_E f_n \to \int_E f$.

Proof.

Example. Determine $\int_{[0,1]} \varphi$ for the Cantor-Lebesgue function $\varphi : [0,1] \to [0,1]$.

4.3 The Lebesgue Integral of a Nonnegative Function

Definition. Let $f : E \to \overline{\mathbf{R}}$ be measurable. We say that f *vanishes outside of a set of finite measure* if there exists a set $E_0 \subseteq E$ such that $mE_0 < \infty$ and $f|_{E-E_0} = 0$. We also say that f *has finite support* and define the *support of* f as

$$\operatorname{supp} f = \{x \in E \mid f(x) \neq 0\}$$

If $f : E \to \mathbf{R}$ is measurable, bounded and has finite support, we define $\int_E f = \int_{E_0} f$, where E_0 is any set so that $mE_0 < \infty$ and $f|_{E-E_0} = 0$ (that is, $\operatorname{supp} f \subseteq E_0$). This is well-defined: it does not depend on the choice of E_0.

More generally, we can define:

Definition. Let $f : E \to \overline{\mathbf{R}}$ be measurable, $f \geq 0$. We define

$$\int_E f = \sup \left\{ \int_E h \mid h \text{ is bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E \right\}$$

Example. Show $\displaystyle\int_{[1,\infty)} \frac{1}{x} = \infty$.

Example. $\displaystyle\int_{[1,\infty)} \frac{1}{x^2}$ is trickier.

Chebyshev's Inequality. Let $f : E \to \overline{\mathbf{R}}$ be measurable, $f \geq 0$. Then for any $\lambda > 0$,

$$m\{x \in E \mid f(x) \geq \lambda\} \leq \frac{1}{\lambda} \int_E f$$

Proof.

Proposition 4.9. Let $f : E \to \overline{\mathbf{R}}$ be measurable, $f \geq 0$. Then

$$\int_E f = 0 \text{ if and only if } f = 0 \text{ a.e. on } E$$

Proof. Similar to a homework problem in section 4.2.

Theorem 4.10. Let $f, g : E \to \overline{\mathbf{R}}$ be measurable, $f, g \geq 0$. Then for every $\alpha, \beta > 0$ we have

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g \qquad \text{if } f \leq g \text{ on } E, \text{ then } \int_E f \leq \int_E g$$

Proof.

Theorem 4.11. Let $f : E \to \overline{\mathbf{R}}$ be measurable, $f \geq 0$, $A, B \subseteq E$ measurable and disjoint. Then

$$\int_{A\cup B} f = \int_A f + \int_B f \qquad \text{If } E_0 \subseteq E, mE_0 = 0, \text{ then } \int_E f = \int_{E-E_0} f$$

Proof. Depends on linearity and $\displaystyle\int_A f = \int f\chi_A$, so it is same as before.

Fatou's Lemma. For every $n \in \mathbf{N}$, let $f_n : E \to \overline{\mathbf{R}}$ be measurable, $f_n \geq 0$.

$$\text{If } f_n \to f \text{ pointwise a.e. on } E, \text{ then } \int_E f \leq \liminf \int_E f_n$$

Recall that $\liminf a_n = \lim\limits_{n\to\infty} \inf\{a_k \mid k \geq n\}$, $\limsup a_n = \lim\limits_{n\to\infty} \sup\{a_k \mid k \geq n\}$. Note that for every sequence (a_n), $\liminf a_n \leq \limsup a_n$.

Proof.

Example. Let f_n be the piecewise linear functions $f_n : [0,1] \to \mathbf{R}$ passing through points $(0,0)$, $\left(\frac{1}{n}, n\right)$, $\left(\frac{2}{n}, 0\right)$ and $(1,0)$. Then $f_n \to 0$ on $[0,1]$. Verify Fatou's Lemma for this example.

Example. Let $g_n = \begin{cases} \frac{1}{2}\chi_{[n,n+1]}, & \text{if } n \text{ is even} \\ \chi_{[n,n+1]}, & \text{if } n \text{ is odd} \end{cases}$ Then $g_n \to 0$ on $\mathbf{R}$. Verify Fatou's Lemma for this example.

Example. Let $g_n = \begin{cases} \chi_{[n,n+1]}, & \text{if } n \text{ is prime} \\ \frac{1}{n}\chi_{[n,n+1]}, & \text{if } n \text{ is not prime} \end{cases}$ Then $g_n \to 0$ on $\mathbf{R}$. Verify Fatou's Lemma for this example.

Monotone Convergence Theorem. For every $n \in \mathbf{N}$, let $f_n : E \to \overline{\mathbf{R}}$ be measurable, $f_n \geq 0$, and let (f_n) be an increasing sequence.

$$\text{If } f_n \to f \text{ pointwise a.e. on } E\text{, then } \int_E f_n \to \int_E f$$

Proof.

Corollary 4.12. For every $n \in \mathbf{N}$, let $u_n : E \to \overline{\mathbf{R}}$ be measurable, $u_n \geq 0$.

$$\text{If } f = \sum_{n=1}^{\infty} u_n \text{ pointwise a.e. on } E\text{, then } \int_E f = \sum_{n=1}^{\infty} \int_E u_n$$

Definition. A measurable function $f : E \to \overline{\mathbf{R}}$, $f \geq 0$, is said to be *integrable* if $\displaystyle\int_E f < \infty$.

Proposition 4.13. If $f : E \to \overline{\mathbf{R}}$, $f \geq 0$, is integrable, then f is finite a.e. on E.

Proof.

Beppo Levi's Lemma. For every $n \in \mathbf{N}$, let $f_n : E \to \overline{\mathbf{R}}$ be measurable, $f_n \geq 0$, and let (f_n) be an increasing sequence. If the set $\{\int_E f_n \mid n \in \mathbf{N}\}$ is bounded, then (f_n) converges pointwise to a measurable function f that is finite a.e. on E and $\int_E f_n \to \int_E f < \infty$.

Proof.

4.4 The General Lebesgue Integral

Recall the definition from section 3.1: for a function $f : E \to \overline{\mathbf{R}}$, we set

$f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$ which implies $f = f^+ - f^-$, $|f| = f^+ + f^-$

Furthermore, $f^+, f^- \geq 0$ and f is measurable if and only if f^+ and f^- both are.

Proposition 4.14. Let $f : E \to \overline{\mathbf{R}}$ be measurable. Then f^+, f^- are integrable if and only if $|f|$ is integrable.

Proof.

Definition. A measurable function $f : E \to \overline{\mathbf{R}}$ is said to be *integrable over* E if $|f|$ is integrable over E. In this case, we define:

$$\int_E f = \int_E f^+ - \int_E f^-$$

Note. The definition agrees with earlier definitions if f is bounded, measurable with finite support or if $f \geq 0$.

Example. Let $f : [1, \infty) \to \mathbf{R}$, $f(x) = (-1)^n \dfrac{1}{n}$ on $[n, n+1)$. Justify the following properties of this function:

1) $\lim_{b\to\infty} \int_1^b f(x)\, dx$ exists
2) The above is a consequence of ordering rather than that "area enclosed" is finite. Rearrange the constant pieces to get a function g for which $\lim_{b\to\infty} \int_1^b g(x)\, dx = -\infty$. This is the reason why textbooks usually only treat indefinite integrals of positive functions.
3) Under Lebesgue integral definition, f is not integrable since $|f|$ is not integrable, $\int_{[1,\infty)} |f| = \infty$

Proposition 4.15. Let $f : E \to \overline{\mathbf{R}}$ be integrable over E. Then f is finite a.e. on E and $\displaystyle\int_E f = \int_{E-E_0} f$ if $E_0 \subseteq E$ and $mE_0 = 0$.

Proof.

Proposition 4.16 (Integral Comparison Test). Let $f : E \to \overline{\mathbf{R}}$ be measurable and suppose there exists a $g : E \to \overline{\mathbf{R}}$ that is integrable over E and dominates f (that is, $|f| \le g$). Then f is integrable and

$$\left| \int_E f \right| \le \int_E |f|$$

Proof.

Theorem 4.17. Let $f, g : E \to \overline{\mathbf{R}}$ be integrable on E. Then for every $\alpha, \beta \in \mathbf{R}$ we have

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g \qquad \text{if } f \leq g \text{ on } E\text{, then } \int_E f \leq \int_E g$$

Proof. See book

Note. If $f(x), g(x) = \infty$, it's not possible to define $(f - g)(x)$. However, due to integrability of both, f and g are finite a.e. on E, so we can reduce E to a smaller set where f and g are both finite.

Corollary 4.18. Let $f : E \to \overline{\mathbf{R}}$ be integrable over E, $A, B \subseteq E$ measurable and disjoint. Then

$$\int_{A \cup B} f = \int_A f + \int_B f$$

Proof.

The Lebesgue Dominated Convergence Theorem. For every $n \in \mathbf{N}$, let $f_n : E \to \overline{\mathbf{R}}$ be measurable and suppose there is a function $g : E \to \overline{\mathbf{R}}$ that is integrable over E and dominates (f_n) on E (that is, $|f_n| \leq g$ for all $n \in \mathbf{N}$). If $f_n \to f$ pointwise a.e. on E, then f is integrable on E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof.

Theorem 4.19 (General Lebesgue Dominated Convergence Theorem). For every $n \in \mathbf{N}$, let $f_n : E \to \overline{\mathbf{R}}$ be measurable, and let $f_n \to f$ pointwise a.e. on E. Suppose there is a sequence $g_n : E \to \overline{\mathbf{R}}$, $g_n \geq 0$, of measurable functions that converges pointwise a.e. on E to a function $g : E \to \overline{\mathbf{R}}$ and dominates (f_n) on E (that is, $|f_n| \leq g_n$ for all $n \in \mathbf{N}$).

$$\text{If } \lim_{n\to\infty} \int_E g_n = \int_E g < \infty, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. Homework.

4.5 Countable Additivity and Continuity of Integration

Theorem 4.20 (Countable Additivity of Integration). Let $f : E \to \overline{\mathbf{R}}$ be integrable over E, $\{E_n, n \in \mathbf{N}\}$ a family of disjoint measurable sets such that $E = \bigcup_{n=1}^{\infty} E_n$. Then

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$$

Proof.

Theorem 4.21 (Continuity of Integration). Let $f : E \to \overline{\mathbf{R}}$ be integrable over E.

1) If $E_1 \subseteq E_2 \subseteq \cdots \subseteq E$ is an ascending collection of measurable subsets, then

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$$

2) If $E \supseteq E_1 \supseteq E_2 \supseteq \dots$ is a descending collection of measurable subsets, then

$$\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$$

Proof. Homework.

4.6 Uniform Integrability, Vitali Convergence Theorem

Lemma 4.14. Let $mE < \infty$ and $\delta > 0$. Then E is a disjoint union of a finite collection of sets, each with measure less than δ.

Proof.

Proposition 4.23. Let $f : E \to \overline{\mathbf{R}}$ be measurable. If f is integrable over E, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\text{if } A \subseteq E \text{ is any measurable set with } mA < \delta, \text{ then } \int_A f < \varepsilon$$

Conversely, if $mE < \infty$ and for every $\varepsilon > 0$ there is a $\delta > 0$ such that above condition holds, then f is integrable over E.

Proof.

Definition. A family $\mathcal{F}$ of measurable functions is *uniformly integrable over* E if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $f \in \mathcal{F}$, if $A \subseteq E$ is measurable and $mA < \delta$, then $\int_A |f| < \varepsilon$.

Example. Lemma 4.23 says if f is integrable over E and $mE < \infty$, then $\{f\}$ is uniformly integrable.

Example. A finite collection of integrable functions is uniformly integrable.

Example. A collection $\mathcal{F}$ dominated by an integrable function $g \geq 0$ is always uniformly integrable.

Example. Consider $f_n : [0,1] \to \mathbf{R}$, $f_n(x) = n$. Then $\{f_n,\, n \in \mathbf{N}\}$ is not uniformly integrable.

Example. Consider the piecewise linear functions $f_n : [0,1] \to \mathbf{R}$ passing through points $(0,0)$, $\left(\frac{1}{n}, n\right)$, $\left(\frac{2}{n}, 0\right)$ and $(1,0)$. Although $\{\int_{[0,1]} f_n, n \in \mathbf{N}\}$ is a bounded set, this collection is not uniformly integrable,

The Vitali Convergence Theorem. Let $mE < \infty$ and suppose $\{f_n : E \to \overline{\mathbf{R}}, n \in \mathbf{N}\}$ is uniformly integrable over E. If $f_n \to f$ pointwise a.e. on E, then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof.

Theorem 4.26. Let $mE < \infty$ and suppose for every $n \in \mathbf{N}$, $h_n : E \to \overline{\mathbf{R}}$, $h_n \geq 0$, is integrable over E and that $h_n \to 0$ a.e. on E. Then

$$\int_E h_n \to 0 \text{ if and only if } \{h_n,\, n \in \mathbf{N}\} \text{ is uniformly integrable over } E$$

Proof.