3.1 Sums, Products and Compositions of LMF

Let $E \subseteq \mathbf{R}$ be measurable, $f : E \to \overline{\mathbf{R}}$ a function, where $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty, -\infty\}$.

Recall that a property holds almost everywhere ("a.e.") if it holds on $E - E_0$, where $E_0 \subseteq E$, $mE_0 = 0$.

Proposition 3.1. Let E be measurable, $f : E \to \overline{\mathbf{R}}$ a function. The following are equivalent

1) For every $c \in \mathbf{R}$, $\{x \in E \mid f(x) > c\}$ is measurable.
2) For every $c \in \mathbf{R}$, $\{x \in E \mid f(x) \geq c\}$ is measurable.
3) For every $c \in \mathbf{R}$, $\{x \in E \mid f(x) < c\}$ is measurable.
4) For every $c \in \mathbf{R}$, $\{x \in E \mid f(x) \leq c\}$ is measurable.

Any of these implies that for every $c \in \mathbf{R}$, $\{x \in E \mid f(x) = c\}$ is measurable.

Proof.

Definition. A function $f : E \to \overline{\mathbf{R}}$, where E is measurable, is said to be *(Lebesgue) measurable* if it satisfies any of the conditions 1–4 in Proposition 3.1.

Note. $E \subseteq \mathbf{R}$ is measurable if and only if the *characteristic function* $\chi_E : \mathbf{R} \to \mathbf{R}$, $\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$ is measurable. This means there exist nonmeasurable functions.

Proposition 3.2. Let $f : E \to \mathbf{R}$ be a function, where E is measurable. Then f is measurable if and only if for every open set $U \subseteq \mathbf{R}$, the set $f^{-1}(U)$ is measurable.

Proof.

Proposition 3.3. Let $f : E \to \mathbf{R}$ be a continuous function, where E is measurable. Then f is measurable.

Proof.

Proposition 3.4. Let $f : I \to \mathbf{R}$ be a monotone function, where I is an interval. Then f is measurable.

Proof. Homework.

Proposition 3.5. Let $f : E \to \overline{\mathbf{R}}$ be a function, where E is measurable.

1) If f is measurable on E and $f = g$ a.e. on E, then g is measurable.

2) If $D \subseteq E$ is measurable, then f is measurable on E if and only if the restrictions $f|_D$ and $f|_{E-D}$ are measurable on their domains.

Theorem 3.6. Let $f, g : E \to \overline{\mathbf{R}}$ be functions, where E is measurable, such that $f, g \neq \pm\infty$ a.e. on E. Then for every $\alpha, \beta \in \mathbf{R}$ we have

$\alpha f + \beta g$ is measurable on E $\qquad\qquad$ fg is measurable on E

Note. If $f(x_0) = \infty$ and $g(x_0) = -\infty$, then $(f+g)(x_0)$ is not defined. This is why we require that $f, g \neq \pm\infty$ a.e. on E. On the set $F \subseteq E$ where f and g are both finite $f+g$ is defined and $m(E-F) = 0$, so by Proposition 3.5 $f+g$ can be defined however we like on $E-F$, and $f+g$ will still be measurable.

Proof.

Proposition 3.7. Let $f : E \to \mathbf{R}$ be a measurable function, where E is measurable, and let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Then $f \circ g$ is a measurable function.

Proof.

Note. Proposition 3.7 implies that $|f|$, $|f|^p$ are measurable if f is.

Example. In general, the composite of measurable functions need not be measurable. Verify this for the composite $\chi_A \circ \psi^{-1}$, where $\psi : \mathbf{R} \to \mathbf{R}$ is the Cantor-Lebesgue function modification $\psi : [0,1] \to [0,2]$, extended to a continuous and strictly increasing function from $\mathbf{R}$ onto $\mathbf{R}$, and A is a measurable set such that $\psi(A)$ is nonmeasurable.

Definition. If $f_1, \ldots, f_n : E \to \overline{\mathbf{R}}$, we define the functions $\min\{f_1, \ldots, f_n\} : E \to \overline{\mathbf{R}}$, $\max\{f_1, \ldots, f_n\} : E \to \overline{\mathbf{R}}$ as

$$\min\{f_1, \ldots, f_n\}(x) = \min\{f_1(x), \ldots, f_n(x)\} \quad \max\{f_1, \ldots, f_n\}(x) = \max\{f_1(x), \ldots, f_n(x)\}$$

Proposition 3.8. If $f_1, \ldots, f_n$ are measurable, so are $\min\{f_1, \ldots, f_n\}$ and $\max\{f_1, \ldots, f_n\}$.

Proof.

Definition. For a function $f : E \to \overline{\mathbf{R}}$,
we define: $f^+(x) = \max\{f(x), 0\}$

$f^-(x) = \max\{-f(x), 0\}$

$|f(x)| = \max\{f(x), -f(x)\}$

$f = f^+ - f^-$ and $|f| = f^+ + f^-$.

f is measurable if and only if f^+ and f^- both are.

3.2 Pointwise Limits and Simple Approximation

The phrase "f_n converges to f" may have several meanings:

Definition. Let $f_n : E \to \overline{\mathbf{R}}$ be a sequence of functions, $A \subseteq E$. We say f_n *converges to* f

1) *pointwise on* A, if $\lim f_n(x) = f(x)$ for all $x \in A$.
2) *pointwise a.e. on* A, if $\lim f_n(x) = f(x)$ for all $x \in A - B$, where $mB = 0$.
3) *uniformly on* A if for every $\varepsilon > 0$ there is a $K \in \mathbf{N}$ such that $|f(x) - f_n(x)| < \varepsilon$ for all $x \in A$ and $n \geq K$.

(Note that uniform convergence only makes sense if $f_n : E \to \mathbf{R}$ for all $n \in \mathbf{N}$.)

Example. Verify the type of convergence for the following examples.

$f_n : [0, \infty) \to \overline{\mathbf{R}}$

$$f_n(x) = \begin{cases} \dfrac{n}{x}, & \text{if } x > 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

$f(x) = \infty$ for all x

$f_n \to f$ pointwise on $[0, \infty)$

$f_n : [0, 1] \to \mathbf{R}$

$f_n(x) = x^n$

$f(x) = 0$ for all x

$f_n \to f$ pointwise a.e. on $[0, 1]$

$f_n : [0, 1] \to \mathbf{R}$

$C =$ Cantor set

$$f_n(x) = \begin{cases} x^n, & \text{if } x \in C \\ 1, & \text{if } x \notin C \end{cases}$$

$f(x) = 1$ for all x

$f_n \to f$ pointwise a.e. $[0, 1]$

$f_n : \mathbf{R} \to \mathbf{R}$

$$f_n(x) = \frac{1}{n} \sin x$$

$f(x) = 0$ for all x

$f_n \to f$ uniformly on $\mathbf{R}$

Proposition 3.9. Let $(f_n) : E \to \overline{\mathbf{R}}$ be a sequence of measurable functions such that $f_n \to f$ pointwise a.e. on E. Then f is measurable.

Proof.

Definition. A function $\varphi : E \to \mathbf{R}$, E measurable, is called *simple* if it is measurable and takes only a finite number of values.

Note. If φ takes on distinct values $c_1, \ldots, c_n \in \mathbf{R}$, then

$$\varphi = \sum_{k=1}^{n} c_k \chi_{E_k}, \text{ where } E_k = \{x \in E \mid \varphi(x) = c_k\}, \text{ a measurable subset of } E$$

Conversely, any function of form $\sum_{k=1}^{n} c_k \chi_{E_k}$ where E_k, $k = 1, \ldots, n$, is measurable, is simple.

The Simple Approximation Lemma. Let $f : E \to \mathbf{R}$ be measurable and bounded on E. Then for every $\varepsilon > 0$ there exist simple functions $\varphi_\varepsilon, \psi_\varepsilon : E \to \mathbf{R}$ such that

$$\varphi_\varepsilon \le f \le \psi_\varepsilon \text{ and } 0 \le \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E$$

(Note similarity to squeeze theorem 7.2.3: $f \in \mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there exist functions $\alpha, \omega \in \mathcal{R}[a,b]$ such that $\alpha(x) \le f(x) \le \omega(x)$ on $[a,b]$ and $\int_a^b \omega - \alpha < \varepsilon$.)

Proof.

The Simple Approximation Theorem. Let $f : E \to \overline{\mathbf{R}}$ be a function, E measurable. Then f is measurable if and only if there exists a sequence $(\varphi_n) : E \to \mathbf{R}$ of simple functions such that $\varphi_n \to f$ pointwise on E and $|\varphi_n| < |f|$ for all $n \in \mathbf{N}$. If $f \geq 0$, we may choose (φ_n) to be increasing. Thus, measurable functions are pointwise limits of simple functions.

Proof.

Example. A sequence of simple functions whose limit is the Cantor-Lebesgue function φ.

3.3 Littlewood's Principles, Lusin's, Egoroff's Theorems

J.E. Littlewood summarizes measure theory in his three principles:

1) Every measurable set is nearly a finite union of intervals.
2) Every measurable function is nearly continuous.
3) Every pointwise-convergent sequence of measurable functions converges nearly uniformly.

More precisely, principle 1 is Theorem 2.12:

Let E be measurable and $mE < \infty$. Then for every $\varepsilon > 0$ there is a disjoint collection of open intervals $I_1, \dots, I_n$ such that

$$m^*(E - U) + m^*(U - E) < \varepsilon, \text{ where } U = I_1 \cup \cdots \cup I_n.$$

Principle 3 is

Egoroff's Theorem. Let $mE < \infty$ and let $(f_n) : E \to \overline{\mathbf{R}}$ be a sequence of measurable functions that converges pointwise on E to $f : E \to R$. Then for every $\varepsilon > 0$ there is a closed set $F \subseteq E$ such that $f_n \to f$ uniformly and $m(E - F) < \varepsilon$.

Lemma 3.10. With same assumptions as Egoroff's Theorem, for each $\eta > 0$ and $\delta > 0$ there exists a measurable subset $A \subseteq E$ and an index N such that

$$|f_n - f| < \eta \text{ on } A \text{ for all } n > N \quad \text{and} \quad m(E - A) < \delta$$

Proof of Lemma 3.10.

Proof of Egoroff's Theorem.

Note. Egoroff's Theorem also holds in an a.e. formulation.

Example. Verify Egoroff's Theorem for this sequence of functions. Let $\{q_n, n \in \mathbf{N}\}$ be a sequence listing all rational numbers in $[0, 1]$.

$f : [0, 1] \to [0, 1]$

$$f(x) = \begin{cases} q_n, & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \\ 0, & \text{if } x = 0 \end{cases}$$

$f_n : [0, 1] \to [0, 1]$

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in \left(\frac{1}{n}, 1\right] \\ 0, & \text{if } x \in \left[0, \frac{1}{n}\right] \end{cases}$$

Example. Verify Egoroff's Theorem for this sequence of functions. Let C = Cantor set $= \bigcap_{k=1}^{\infty} C_k$, $U_k = [0,1] - C_k$, $U = \bigcup_{k=1}^{\infty} U_k$, $\varphi_n : U_n \to [0,1]$ the function in the construction of the Cantor-Lebesgue function.

$f_n : [0,1] \to [0,1]$ $\qquad$ $f : [0,1] \to [0,1]$

$$f_n(x) = \begin{cases} \varphi_n(x), & \text{if } x \in U_n \\ 0, & \text{if } x \notin U_n \end{cases} \qquad f(x) = \begin{cases} \varphi(x), & \text{if } x \in U \\ 0, & \text{if } x \in C \end{cases}$$

Proposition 3.11. Let $f : E \to \mathbf{R}$ be a simple function. Then for every $\varepsilon > 0$ there is a continuous function $g : \mathbf{R} \to \mathbf{R}$ and a closed set $F \subseteq E$ such that $f = g$ on F and $m(E - F) < \varepsilon$.

Proof.

Lusin's Theorem. Let $f : E \to \mathbf{R}$ be a measurable function, E measurable. Then for every $\varepsilon > 0$ there is a continuous function $g : \mathbf{R} \to \mathbf{R}$ and a closed set $F \subseteq E$ such that $f = g$ on F and $m(E - F) < \varepsilon$.

Proof.

Example. Verify Lusin's Theorem for the function $f : [0,1] \to \mathbf{R}$, $f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0, & \text{if } x \notin \mathbf{Q} \end{cases}$