2.1 Introduction to Measure Theory

We wish to define a function (measure)

$$m : \{\text{some subsets of } \mathbf{R}\} \to [0, \infty]$$

which captures the idea of "size," which in $\mathbf{R}$ is "length." (If we were working in $\mathbf{R}^2$ or $\mathbf{R}^3$, "size" would be "area" or "volume.")

Definition. Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbf{R}$ that contains all intervals. A function $m : \mathcal{A} \to [0, \infty] = [0, \infty) \cup \{\infty\}$ is called a *Lebesgue measure* if it possesses these properties:

1) *Meausure of an interval is its length.* If I is an interval — (a, b), $[a, b)$, $(a, b]$, $[a, b]$, where open bounds could be ∞ — then $m(I)$ =length of I (possibly ∞)

2) *Measure is translation-invariant.* If $E \in \mathcal{A}$, then for every $y \in \mathbf{R}$, $E + y = \{e + y \mid e \in E\}$ is also in $\mathcal{A}$ and $m(E + y) = m(E)$.

3) *Measure is countably additive over countable disjoint unions.* If $\{E_k, k \in \mathbf{N}\}$ is a disjoint collection of sets in $\mathcal{A}$, then $m\left(\bigcup_{k\in\mathbf{N}} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$ (disjoint collection means for every $i \neq j$, $E_i \cap E_j = \emptyset$).

It turns out, it is not possible to achieve this for $\mathcal{A} = \mathcal{P}(\mathbf{R})$ = all subsets of $\mathbf{R}$, but it is for a smaller collection, a σ-algebra called *Lebesgue measurable* sets, which contain the Borel sets.

To prove existence of such a measure function, we start with a function called *outer measure*.

2.2 Lebesgue Outer Measure

Definition. Let I be an open interval, $I = (a, b)$, where $a \in \{-\infty\} \cup \mathbf{R}$ and $b \in \mathbf{R} \cup \{\infty\}$. The *length of* I, $\ell(I)$, is defined as:

$$\ell(I) = \begin{cases} b - a, & \text{if } a, b \in \mathbf{R} \\ \infty, & \text{if } a = -\infty \text{ or } b = \infty \end{cases}$$

Definition. Let $A \subseteq \mathbf{R}$. The *outer measure of* A, $m^*(A)$ or m^*A, is defined as

$$m^*A = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \mid A \subseteq \bigcup_{k=1}^{\infty} I_k, \quad \begin{array}{l} \text{where } \{I_k, k \in \mathbf{N}\} \text{ is} \\ \text{a cover of } A \text{ by open intervals} \end{array} \right\}$$

Note. 1) $m^*\emptyset = 0$

2) If $A \subseteq B$, then $m^*A \leq m^*B$.

Example. If A is countable, then $m^*A = 0$.

Proposition 2.1. If I is an interval, then $m^*I = \ell(I)$.

Proof.

Proposition 2.2. Outer measure is translation-invariant, that is, for every $A \subseteq \mathbf{R}$, $y \in \mathbf{R}$, $m^*(A + y) = m^*A$.

Proof.

Theorem 2.3. Outer measure is countably subadditive, that is, for every countable collection $\{E_k, k \in \mathbf{N}\}$ of subsets of $\mathbf{R}$, $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^* E_k$.

Proof.

2.3 Lebesgue Measurable Sets

For an outer measure m^*, we know that $m^*(A \cup B) \leq m^*A + m^*B$ holds. However, there exist disjoint sets for which

$$m^*(A \cup B) < m^*A + m^*B, \text{ which is not desirable for a measure.}$$

Setting $E = A$, $C = A \cup B$, this can be rewritten as

$$m^*(C) < m^*(E \cap C) + m^*(E^c \cap C), \text{ again, not desirable for a measure.}$$

Definition. A set E is *measurable* if for any set A

$$m^*A = m^*(A \cap E) + m^*(A \cap E^c)$$

It immediately follows that if one of A, B is measurable and A, B are disjoint, then $m^*(A \cup B) = m^*A + m^*B$.

Note.

1) Since $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$, to show E is measurable we only need to show the opposite inequality: $m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A)$.
2) E is measurable if and only if E^c is measurable.
3) $\emptyset$ and $\mathbf{R}$ are measurable.

Proposition 2.4. Any set of outer measure zero is measurable. In particular, all countable sets are measurable.

Proof.

Proposition 2.5. The union of a finite collection of sets is measurable.

Proof.

Proposition 2.5 shows that the collection of measurable sets is an algebra (defined like a σ-algebra, except with closure with respect to finite unions instead of countable).

Proposition 2.6. Let $E_1, \ldots, E_n$ be disjoint measurable sets and $A \subseteq \mathbf{R}$. Then

$$m^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) = \sum_{k=1}^{n} m^*(A \cap E_k) \quad \text{and} \quad m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^* E_k$$

Proof.

Proposition 2.7. The union of a countable collection of measurable sets is measurable. The collection of measurable sets is a σ-algebra.

Proof.

Proposition 2.8. Every interval is measurable.

Proof.

Note.

1) The collection of measurable sets is a σ-algebra.
2) Every open set is measurable — it is a countable union of open intervals.
3) Every closed set is measurable — it is a complement of an open set.
4) Every F_σ and G_δ set is measurable — they are intersections and unions of countable collections of closed and open sets
5) Every Borel set is measurable — it is in the smallest σ-algebra that contains open sets and measurable sets are one σ-algebra that contains open sets.

Thus we have proved:

Theorem 2.9. The collection of measurable sets is a σ-algebra that contains the Borel sets.

Proposition 2.10. The translate of a measurable set is measurable.

Proof.

2.4 Outer/Inner Approx. of Lebesgue Measurable Sets

Let A be measurable, $m^*A < \infty$. Then for any set $B \supseteq A$ we have

$$m^*(B - A) = m^*B - m * A \qquad \text{the excision property}$$

Theorem 2.11. Let E be any set. Then measurability of E is equivalent to any of the following four conditions.

Outer approximation by open and G_δ sets:

1) For every $\varepsilon > 0$ there is an open set $U \supseteq E$ such that $m^*(U - E) < \varepsilon$.
2) There exists a G_δ-set $G \supseteq E$ such that $m^*(G - E) = 0$.

Inner approximation by closed and F_σ sets:

3) For every $\varepsilon > 0$ there is a closed set $F \subseteq E$ such that $m^*(E - F) < \varepsilon$.
4) There exists an F_σ-set $F \subseteq E$ such that $m^*(E - F) = 0$.

Proof.

Note. The theorem implies that measurable sets have form $E = G - Y = F \cup Z$ where G is a G_δ-set, F is an F_σ-set and Y and Z are sets of measure zero.

Note. For any set E there is an open set $U = \bigcup_{k=1}^{\infty} I_k$ such that $m^*U < m^*E + \varepsilon$, so assuming m^*E is finite, $m^*U - m^*E < \varepsilon$, but this does not mean that $m^*(U - E) < \varepsilon$ because $m^*(U - E) = m^*U - m^*E$ is valid only for measurable sets E.

Theorem 2.12. Let E be measurable and $m^*E < \infty$. Then for every $\varepsilon > 0$ there is a disjoint collection of open intervals $I_1, \ldots, I_n$ such that

$$m^*(E - U) + m^*(U - E) < \varepsilon, \text{ where } U = I_1 \cup \cdots \cup I_n.$$

Proof.

2.5 Countable Additivity, Continuity, Borel-Cantelli

Definition. Let $\mathcal{M}$ be the σ-algebra of measurable subsets of $\mathbf{R}$.
The function $m : \mathcal{M} \to [0, \infty]$ defined by $mE = m^*E$ is called *the Lebesgue measure.*

Theorem 2.13. Lebesgue measure is countably additive, that is, if $\{E_k, k \in \mathbf{N}\}$ is a disjoint collection of measurable sets, then

$$\bigcup_{k=1}^{\infty} E_k \text{ is measurable and } m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} mE_k$$

Proof.

Theorem 2.14. The function $m : \mathcal{M} \to [0, \infty]$ is a Lebesgue measure as defined in 2.1 (assigns length to any interval, is translation invariant and countably additive).

Theorem 2.15 (continuity of measure).

1) If $\{A_k, k \in \mathbf{N}\}$ is an ascending collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} mA_k$$

2) If $\{B_k, k \in \mathbf{N}\}$ is a descending collection of measurable sets and $mB_1 < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} mB_k$$

Proof.

Definition. If E is measurable, we say *a property $\mathcal{P}$ holds almost everywhere on E (a.e. on E, holds for almost all $x \in E$)* if there is a subset $E_0 \subseteq E$ such that $mE_0 = 0$ and $\mathcal{P}$ holds for all $x \in E - E_0$.

The Borel Cantelli Lemma. Let $\{E_k, k \in \mathbf{N}\}$ be a collection of measurable sets satisfying $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbf{R}$ belong to at most finitely many of the sets E_k.

Proof.

2.6 Nonmeasurable sets

Lemma 2.16. Let $E \subseteq \mathbf{R}$ be bounded and measurable. Suppose there is a bounded, countably infinite (= denumerable) set of numbers Λ such that the collection $\{E+\lambda, \lambda \in \Lambda\}$ is disjoint. Then $mE = 0$.

Proof.

Example. Let $E = \left\{\frac{m}{2^n} \mid m, n \in \mathbf{N}, m \text{ odd}\right\} \cap [0,1]$ and $\Lambda = \left\{\frac{1}{3^k} \mid k \geq 1\right\}$. E is measurable since it is countable and Λ is countably infinite. Then the collection $\{E+\lambda, \lambda \in \Lambda\}$ is disjoint.

Definition. On an $E \subseteq \mathbf{R}$ define the relation of *rational equivalence*: x and y are *rationally equivalent* ($x \sim y$) if $x - y \in \mathbf{Q}$. This is an equivalence relation.

The equivalence relation partitions E into equivalence classes. Let C_E be a set of class representatives (choice set), one element from every class.

Then C_E satisfies:

i) If $a, b \in C_E$, then $a - b \notin \mathbf{Q}$.
ii) For every $x \in E$ there exists a $c \in C_E$ such that $x \sim c$, so $x = c + q$ for some $q \in \mathbf{Q}$.

This implies that for any subset $\Lambda \subseteq \mathbf{Q}$, the collection $\{C_E + \lambda, \lambda \in \Lambda\}$ is disjoint.

Example. Let $E = [0, 1]$. Then C_E consist of one rational number and a collection of irrational ones. Show that C_E is uncountable.

Vitali's Theorem 2.17. Any set of real numbers E with $m^*E > 0$ contains a subset that is not measurable.

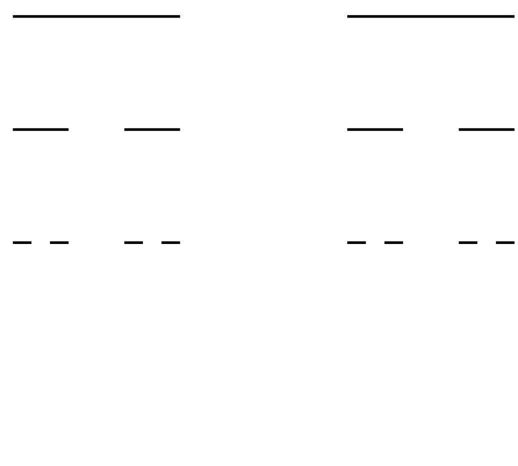
Theorem 2.18. There exist disjoint sets of real numbers A and B such that

$$m^*(A \cup B) < m^*A + m^*B$$

Proof.

2.7 The Cantor Set and the Cantor-Lebesgue Function

Definition. We construct the Cantor set. Let $I = [0,1]$ and define $\{C_k, k \in \mathbf{N}\}$ recursively:



$C_1 = I - \left(\frac{1}{3}, \frac{2}{3}\right)$

$C_2 = C_1 - \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right)$

$C_3 = C_2 - \left(\left(\frac{1}{27}, \frac{2}{27}\right) \cup \left(\frac{7}{27}, \frac{8}{27}\right) \cup \left(\frac{19}{27}, \frac{20}{27}\right) \cup \left(\frac{25}{27}, \frac{26}{27}\right)\right)$

$C_{k+1} = C_k - (\text{open middle thirds of intervals in } C_k)$

We see that:

- — C_k is a union of 2^k disjoint intervals of length $\frac{1}{3^k}$.
- — $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$, so $\{C_k, k \in \mathbf{N}\}$ is a descending sequence of closed sets.

We define *the Cantor set* C as $C = \displaystyle\bigcap_{k=1}^{\infty} C_k$.

Note that C is nonempty for any of these reasons:

- — the Nested Set Theorem: intersection of a bounded descending collection of closed sets is nonempty.
- — $0, 1 \in C_k$ for every k, so $0, 1 \in C$.
- — More generally, if a is an endpoint of one of the intervals in C_k, then $a \in C_{k+1}$, and thus $a \in C_n$ for all $n \geq k$, it follows that $a \in C$.

Proposition 2.19. The Cantor set C is a closed uncountable set of measure 0.

Proof.

Note. The Cantor set is an example of an uncountable set of measure 0 (so far, only such examples have been countable sets).

Note. The following holds for every $x \in [0,1]$:

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \text{ for some } a_k \in \{0,1,2\} \qquad\qquad x \in C \Longleftrightarrow x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \text{ for some } a_k \in \{0,2\}$$

The first statement is essentially writing x as a decimal with base 3. The second statement is not hard to see because $a_k = 0, 2$ if and only if $x \in C_k$. (Also note that endpoints of the intervals in C_k are sums where a_k is constant, 0 or 2, from some index on.) Since this representation of elements of the Cantor set is unique, it gives a bijection between C and sequences of 0s and 2s, of which there are uncountably many.

Definition. Let $U_k = I - C_k$.

Since C_k consists of 2^k disjoint closed intervals, U_k consists of $2^k - 1$ disjoint open intervals. Define $\varphi_k(x) = \frac{i}{2^k}$ if x is in the i-th interval of U_k. Thus, φ_k is constant on every interval of U_k and takes on the values $\frac{1}{2^k}, \frac{2}{2^k}, \ldots, \frac{2^k-1}{2^k}$, which can be rewritten as $\frac{2}{2^{k+1}}, \frac{4}{2^{k+1}}, \ldots, \frac{2(2^k-1)}{2^{k+1}}$. Because the $2^{k+1}-1$ intervals of U_{k+1} are obtained from the 2^k-1 intervals of U_k by inserting a new interval between every two intervals of U_k and two at the end ($2^k-1+2^k-1+1 = 2^{k+1}-1$), the i-th interval of U_k becomes the $2i$-th interval of U_{k+1}, so $\varphi_k(x) = \varphi_{k+1}(x)$ for $x \in U_k$.

This justifies that the following function φ is well-defined:

$$U = \bigcup_{k=1}^{\infty} U_k = \bigcup_{k=1}^{\infty} (I - C_k) = I - \bigcap_{k=1}^{\infty} C_k = I - C, \quad \varphi : U \to [0,1], \quad \varphi(x) = \varphi_k(x) \text{ if } x \in U_k$$

and we extend this function to the *Cantor-Lebesgue function*

$$\varphi : [0,1] \to [0,1], \quad \varphi(x) = \begin{cases} \varphi(x), & \text{if } x \in U \\ 0, & \text{if } x = 0 \\ \sup\{\varphi(t) \mid t \in U, t < x\}, & \text{if } x \in C - \{0\} \end{cases}$$

Proposition 2.20. The Cantor-Lebesgue function $\varphi : [0,1] \to [0,1]$ is increasing, continuous, surjective and $\varphi' = 0$ on U, where $U = [0,1] - C$ and $mU = 1$.

Proof.

Proposition 2.21. Let $\varphi : [0, 1] \to [0, 1]$ be the Cantor-Lebesgue function and set $\psi : [0, 1] \to \mathbf{R}$, $\psi(x) = x + \varphi(x)$. Then ψ is a strictly increasing and continuous function, and $\psi([0, 1]) = [0, 2]$. Furthermore,

i) $\psi(C)$ is a measurable set of positive measure.

ii) ψ maps a measurable subset of C onto a nonmeasurable set.

Proof.

Proposition 2.22. There exists a measurable subset of C that is not a Borel set.

Proof.