Trigonometry — Exam 3
MAT 145, Spring 2025— D. Ivanšić

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Show all your work!

$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$	$\sin(2u) = 2\sin u \cos u$
$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$	$\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$
$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$	$\tan(2u) = \frac{2\tan u}{1-\tan^2 u}$
$\cos^2 \frac{u}{2} = \frac{1+\cos u}{2}$ $\quad \sin^2 \frac{u}{2} = \frac{1-\cos u}{2}$	$\tan^2 \frac{u}{2} = \frac{1-\cos u}{1+\cos u}$

1. (6pts) Solve the triangle: $b = 5$, $c = 3$, $C = 40°$.

$$\frac{3}{\sin 40^\circ} = \frac{5}{\sin B}$$

$$\sin B = \frac{5 \sin 40^\circ}{3} = 1.07. > 1 \text{ so no sol.}$$

$$3\sin B = 5\sin 40^\circ$$

2. (12pts) Solve the triangle: $a = 4$, $c = 7$, $B = 112°$

$$b^2 = 4^2 + 7^2 - 2\cdot 4\cdot 7 \cos 112^\circ$$

$$b^2 = 16 + 49 - 56\cos 112^\circ = 85.97..$$

$$b = 9.2724306$$

$$\cos A = \frac{9.27..^2 + 7^2 - 4^2}{2\cdot 9.27..\cdot 7} = 0.916526$$

$$A = \arccos 0.916526 = 23.576582$$

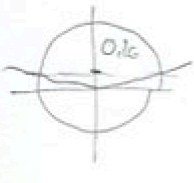
$$C = 180^\circ - (112^\circ + 23.57..^\circ)$$

$$= 44.423418$$

3. (14pts) Solve the triangle: $a = 7$, $c = 2$, $A = 67°$.

$$\frac{7}{\sin 67^\circ} = \frac{2}{\sin C}$$

$$7\sin C = 2\sin 67^\circ$$



$$\sin C = \frac{2\sin 67^\circ}{7} = 0.263001$$

$$C = \arcsin 0.26.. = 15.248229$$

or $C = 180^\circ - \arcsin 0.26.. \doteq 164.75..$

$$\underline{+\ 67^\circ} \quad \text{too big}$$

$$> 180^\circ$$

$$B = 180 - (67^\circ + 15.248..)$$

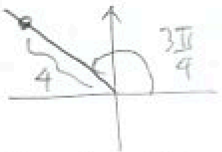
$$= 97.751771$$

$$\frac{7}{\sin 67^\circ} = \frac{b}{\sin 97.75..}$$

$$b = \frac{7\sin 97.75..}{\sin 67^\circ} = 7.53503$$

4. (8pts) Draw points with the following polar coordinates. Then convert them into rectangular coordinates. Give exact answers — do not use the calculator.

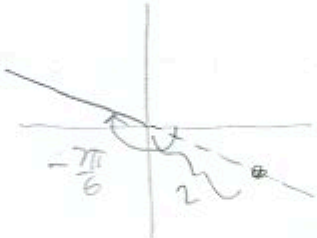
$(r,\theta) = \left(4, \frac{3\pi}{4}\right)$



$x = 4\cos\frac{3\pi}{4} = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$

$y = 4\sin\frac{3\pi}{4} = 4\cdot\frac{\sqrt{2}}{2} = 2\sqrt{2}$

$(r,\theta) = \left(-2, -\frac{7\pi}{6}\right)$



$x = -2\cos\left(-\frac{7\pi}{6}\right) = -2\cdot\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

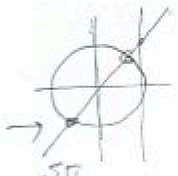
$y = -2\sin\left(-\frac{7\pi}{6}\right) = -2\cdot\frac{1}{2} = -1$

5. (12pts) Convert the following rectangular coordinates into polar coordinates. Draw a picture to make sure you have the correct θ. For each point, give three answers in polar coordinates, at least one of which has a negative r. Give exact answers — do not use the calculator.

$(x, y) = (-5, -5)$

$r = \sqrt{(-5)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$

$\tan\theta = \frac{-5}{-5} = 1$



in correct quad

$\theta = \frac{5\pi}{4}$

$\left(5\sqrt{2}, \frac{5\pi}{4}\right)$

$\left(5\sqrt{2}, -\frac{3\pi}{4}\right)$

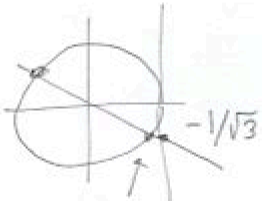
$\left(-5\sqrt{2}, \frac{\pi}{4}\right)$

$(x, y) = (3\sqrt{3}, -3)$

$r = \sqrt{(3\sqrt{3})^2 + (-3)^2} = \sqrt{27+9} = 6$

$\tan\theta = \frac{-3}{3\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$\theta = -\frac{\pi}{6}$



$\left(6, -\frac{\pi}{6}\right)$

$\left(6, \frac{11\pi}{6}\right)$

$\left(-6, \frac{5\pi}{6}\right)$

6. (8pts) Convert to a polar equation. Answer should be solved for r.

$x^2 - 2x + y^2 = \cancel{4}\ 0$

$(r\cos\theta)^2 - 2r\cos\theta + (r\sin\theta)^2 = 0$

$r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta = 0$

$r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta = 0$

$r^2 - 2r\cos\theta = 0$

$r(r - 2\cos\theta) = 0 \quad \div r$

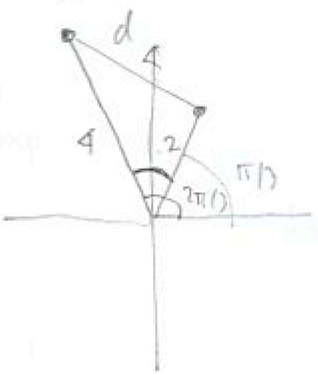
$r - 2\cos\theta = 0$

$r = 2\cos\theta$

7. (8pts) Determine the distance between points given in **polar coordinates**: $B = \left(4, \frac{2\pi}{3}\right)$, $C = \left(2, \frac{\pi}{3}\right)$.

a) Draw the picture.

b) Find the exact distance from B to C (do not use the calculator).

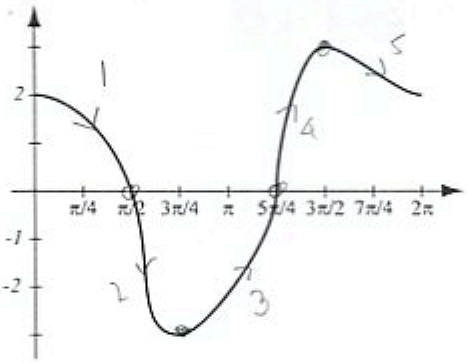


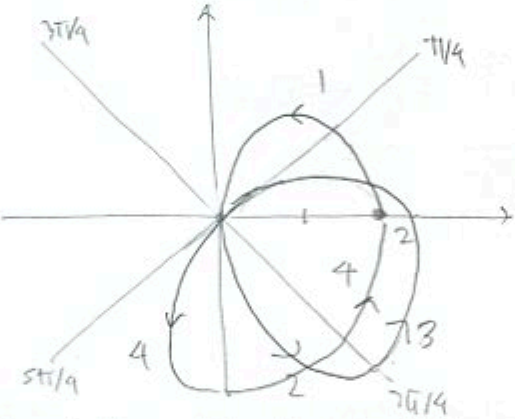
Law of cosines: $d^2 = 2^2 + 4^2 - 2\cdot 2\cdot 4\cos\left(\frac{2\pi}{3} - \frac{\pi}{3}\right)$

$d^2 = 20 - 16\underbrace{\cos\frac{\pi}{3}}_{=\frac{1}{2}} = 12$

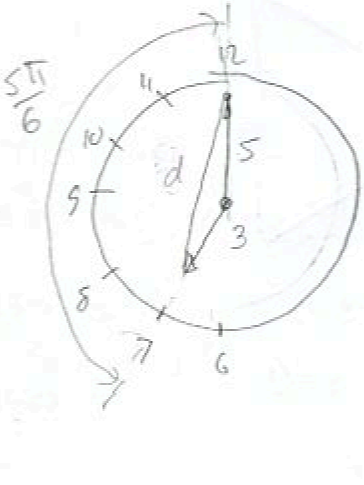
$d = \sqrt{12} = 2\sqrt{3}$

8. (8pts) Below is the graph of the function $r = f(\theta)$ in rectangular r-θ coordinates. Use the graph to draw the graph of $r = f(\theta)$ in polar coordinates, indicating corresponding parts of the graphs.





9. (10pts) If the long hand of the clock has length 5 in and the short hand 3 in, what is the distance between the tips of the hands at 7 o'clock?



Law of cosines:

$d^2 = 5^2 + 3^2 - 2\cdot 3\cdot 5\cos\frac{5\pi}{6}$

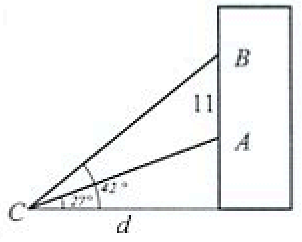
$= 34 - 30\left(-\frac{\sqrt{3}}{2}\right) = 34 + 15\sqrt{3}$

$d = \sqrt{34 + 15\sqrt{3}} \approx 7.744725$

10. (14pts) To determine distance d to a building, sightings of points A and B on the building are made and they stand at angles of elevation $27°$ and $42°$. It is known that the distance from A to B is 11 meters.

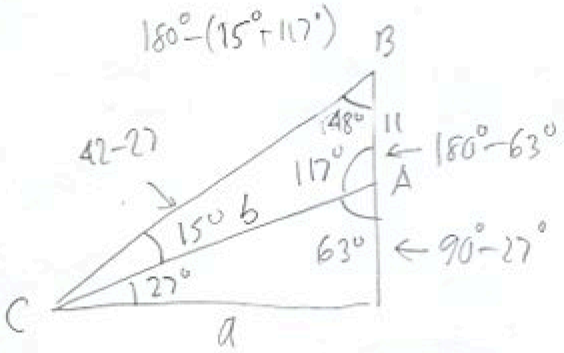
a) Determine angles in the triangle ABC.

b) Find the distance to the building d.



$$\frac{b}{\sin 48°} = \frac{11}{\sin 15°}$$

$$b = \frac{11 \sin 48°}{\sin 15°} = 31.584202$$

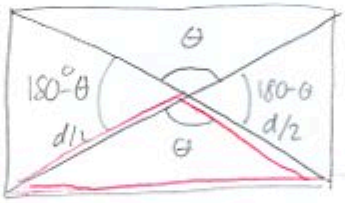


$$\frac{d}{b} = \cos 27°$$

$$d = b\cos 27° = 31.58\ldots \cdot \cos 27°$$

$$= 28.14173\ldots \text{ meters}$$

Bonus. (10pts) Show that the area of a rectangle is one-half of product of lengths of diagonals times the sine of the angle between them.



Diagonals have same length

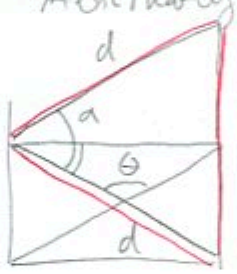
Area of a triangle

$$= \frac{1}{2} \cdot \frac{d}{2} \cdot \frac{d}{2} \sin\theta = \frac{d^2 \sin\theta}{8}$$

The four triangles have equal area,

so area of rectangle $= 4 \cdot \frac{d^2 \sin\theta}{8} = \frac{1}{2} d^2 \sin\theta$

Alternately



Area of rectangle = area of triangle $= \frac{1}{2} d^2 \sin\alpha$

$$\frac{\alpha}{2} + \frac{\alpha}{2} + \theta = 180°$$

$$\alpha = 180 - \theta$$

$$= \frac{1}{2} d^2 \sin(180 - \theta)$$

$$= \frac{1}{2} d^2 \sin\theta$$