Trigonometry — Exam 2	
MAT 145, Spring 2025— D. Ivanšić	

Name: Saul Ocean

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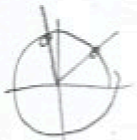
$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$	$\sin(2u) = 2 \sin u \cos u$
$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$	$\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$
$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$	$\tan(2u) = \frac{2 \tan u}{1 - \tan^2 u}$
$\cos^2 \frac{u}{2} = \frac{1+\cos u}{2}$ $\quad \sin^2 \frac{u}{2} = \frac{1-\cos u}{2}$	$\tan^2 \frac{u}{2} = \frac{1-\cos u}{1+\cos u}$

1. (16pts) Use an identity (sum, difference, half- or double-angle) to find the exact values of the trigonometric functions below (do not use the calculator).

$\sin \frac{11\pi}{12} = \sin \frac{(8+3)\pi}{12} = \sin\left(\frac{8\pi}{12} + \frac{3\pi}{12}\right) = \sin\left(\frac{2\pi}{3} + \frac{\pi}{4}\right) = \sin \frac{2\pi}{3} \cos \frac{\pi}{4} + \cos \frac{2\pi}{3} \sin \frac{\pi}{4}$



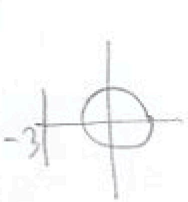
$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

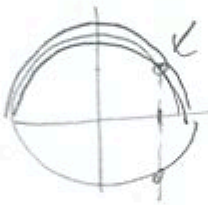
$\cos 67.5° = \cos \frac{135°}{2} = \pm\sqrt{\frac{1 + \cos 135°}{2}} = \pm\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2} \cdot \frac{2}{2}} = \pm\sqrt{\frac{2 - \sqrt{2}}{4}} = \pm \frac{\sqrt{2 - \sqrt{2}}}{2}$

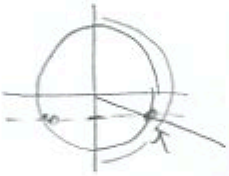
since $0 < 67.5° < 90°$, $\cos 67.5 = \frac{\sqrt{2 - \sqrt{2}}}{2}$

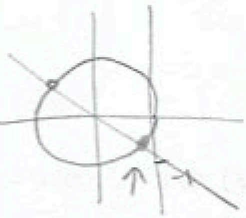
2. (8pts) Without using the calculator, find the exact values (in radians) of the following expressions. Draw the unit circle to help you.

$\arccos(-3) =$ not def. $\quad \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6} \quad \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \quad \arctan(-1) = -\frac{\pi}{4}$



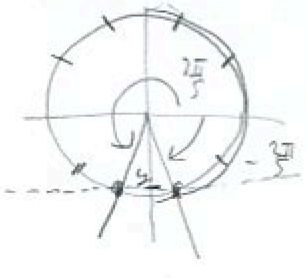






3. (6pts) Find the exact value of the expressions (do not use the calculator). For one of them, you will need a picture.

$\tan(\arctan 3.7) = 3.7 \qquad \arcsin\left(\sin \frac{7\pi}{5}\right) = \arcsin y = -\frac{2\pi}{5}$



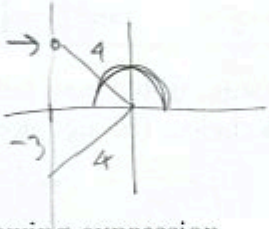
4. (6pts) Find the exact value of the expression (do not use the calculator). Draw the appropriate picture.

$$\tan\left(\arccos\left(-\frac{3}{4}\right)\right) = \quad \tan\theta = \frac{\sqrt{7}}{-3} = -\frac{\sqrt{7}}{3}$$

$\theta = \arccos\left(-\frac{3}{4}\right)$

$\cos\theta = -\frac{3}{4}$, θ in $[0, \pi)$

$= \frac{x}{r} = \frac{-3}{4}$



$(-3)^2 + y^2 = 4^2$

$9 + y^2 = 16$

$y^2 = 7 \quad y = \pm\sqrt{7}, \; y = \sqrt{7}$

5. (8pts) Use identities to simplify the following expression.

$$\left(\sin\left(\frac{\pi}{2}-\theta\right)+\sin\theta\right)\left(\cos\theta-\cos\left(\frac{\pi}{2}-\theta\right)\right) = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$$

$$= \cos^2\theta - \sin^2\theta = \cos(2\theta)$$

Show the identities:

6. (8pts) $\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$

$$\sin\theta(\csc\theta - \sin\theta) = \sin\theta\csc\theta - \sin^2\theta$$

$$= \cancel{\sin\theta}\,\frac{1}{\cancel{\sin\theta}} - \sin^2\theta$$

$$= 1 - \sin^2\theta = \cos^2\theta$$

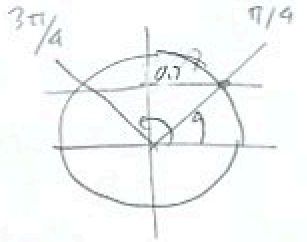
7. (10pts) $\tan\theta + \cot\theta = \dfrac{2}{\sin(2\theta)}$

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin\theta\sin\theta + \cos\theta\cdot\cos\theta}{\cos\theta\sin\theta} = \frac{\overbrace{\sin^2\theta + \cos^2\theta}^{=1}}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}\cdot\frac{2}{2} = \frac{2}{2\sin\theta\cos\theta} = \frac{2}{\sin(2\theta)}$$

8. (5pts) Solve the equation in radians (state general solution).

$4\sin\theta - 2\sqrt{2} = 0$



$4\sin\theta = 2\sqrt{2}$

$\sin\theta = \frac{2\sqrt{2}}{4}$

$\sin\theta = \frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{4} + k\cdot 2\pi$

$\theta = \frac{3\pi}{4} + k\cdot 2\pi$

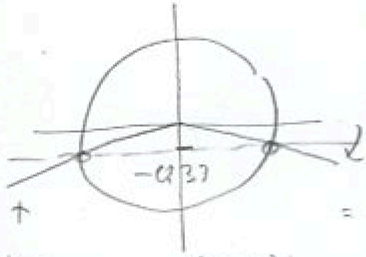
9. (7pts) Use your calculator to solve the equation on the interval $[0°, 360°)$ (answers in degrees). A picture will help.

$\sin\theta = -0.33$

$\arcsin(-0{,}33) = -19.268775$

$360 + \arcsin(-0{,}33) = 340.731225°$

$180° - \arcsin(-0{,}33) = 199.268776°$



$2\ \arcsin(-0{,}33)$

$= \arcsin(-0{,}33) + 360°$

$180 - \arcsin(-0{,}33)$

10. (14pts) Solve the equation in radians.
a) State the general solution.
b) List all the solutions that fall in the interval $[0, 2\pi)$.

$6\cos^2\theta - \cos\theta - 1 = 0$

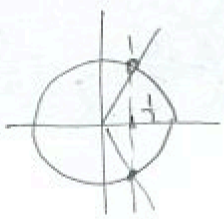
$u = \cos\theta$

$6u^2 - u - 1 = 0$ $1+24$

$u = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot 6\cdot(-1)}}{2\cdot 6}$

$= \frac{1 \pm \sqrt{25}}{12} = \frac{1\pm 5}{12} = \frac{6}{12}, -\frac{4}{12} = \frac{1}{2}, -\frac{1}{3}$

$\cos\theta = \frac{1}{2}$



general

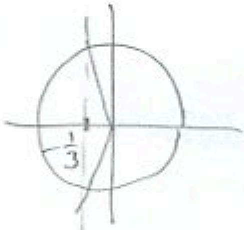
$\theta = \frac{\pi}{3} + k\cdot 2\pi$

$\theta = -\frac{\pi}{3} + k\cdot 2\pi$

In $[0, 2\pi)$

$\frac{\pi}{3}, \frac{5\pi}{3}$

$\cos\theta = -\frac{1}{3}$



$\theta = \arccos(-\frac{1}{3}) + k2\pi$

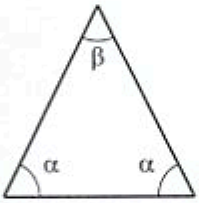
$\theta = -\arccos(-\frac{1}{3}) + k\cdot 2\pi$

$\arccos(-\frac{1}{3})$
$= 1.910633$

$2\pi - \arccos(-\frac{1}{3})$
$= 4.372552$

radians

11. (12pts) An isosceles triangle has two sides of equal length and two angles of same measure. If an isosceles triangle has angles α, α and β, and $\cos\alpha = \frac{1}{3}$, find the exact value of $\cos\beta$ (do not use the calculator).



$\alpha + \alpha + \beta = \pi$

$2\alpha + \beta = \pi$

$\beta = \pi - 2\alpha$

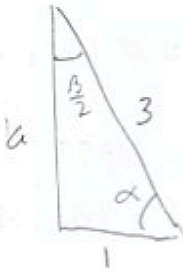
$\cos\beta = \cos(\pi - 2\alpha)$

$= \underbrace{\cos\pi}_{=-1} \cos(2\alpha) + \underbrace{\sin\pi}_{=0} \sin(2\alpha)$

$= -\cos(2\alpha) = -(2\cos^2\alpha - 1)$

$= -\left(2\cdot\left(\frac{1}{3}\right)^2 - 1\right) = -\left(\frac{2}{9} - 1\right) = -\frac{-7}{9} = \frac{7}{9}$

OR:



$a^2 + 1^2 = 3^2$

$a^2 = 8$

$a = \pm\sqrt{8} = \sqrt{8}$

$\cos\frac{\beta}{2} = \frac{\sqrt{8}}{3}$

$\cos\beta = \cos\left(2\cdot\frac{\beta}{2}\right) = 2\cos^2\frac{\beta}{2} - 1 = 2\left(\frac{\sqrt{8}}{3}\right)^2 - 1 = 2\cdot\frac{8}{9} - 1 = \frac{16-9}{9} = \frac{7}{9}$

Bonus. (10pts) Suppose that $\pi < \alpha < \frac{3\pi}{2}$ and $\frac{3\pi}{2} < \beta < 2\pi$ are angles so that $\cos\alpha = -\frac{2}{5}$ and $\cos\beta = \frac{2}{7}$. Find the exact value of $\sin(2\alpha + 2\beta)$.

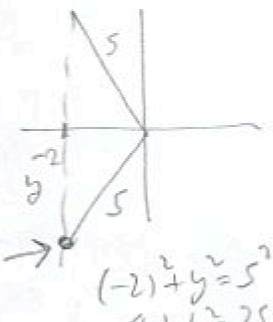
$\sin(2\alpha + 2\beta) = \sin(2(\alpha+\beta)) = 2\sin(\alpha+\beta)\cos(\alpha+\beta)$

$= 2(\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$

$= 2\left(-\frac{\sqrt{21}}{5}\cdot\frac{2}{7} + \left(-\frac{2}{5}\right)\cdot\left(-\frac{3\sqrt{5}}{7}\right)\right)\left(\left(-\frac{2}{5}\right)\frac{2}{7} - \left(-\frac{\sqrt{21}}{5}\right)\left(-\frac{3\sqrt{5}}{7}\right)\right) = 2\,\frac{2\sqrt{21}+6\sqrt{5}}{35}\cdot\frac{-4-3\sqrt{105}}{35}$

$= -\frac{2(2\sqrt{21}+6\sqrt{5})(4+3\sqrt{105})}{35^2}$

$\cos\alpha = -\frac{2}{5} = \frac{-2}{5}$



$\sin\alpha = \frac{-\sqrt{21}}{5}$

$(-2)^2 + y^2 = 5^2$

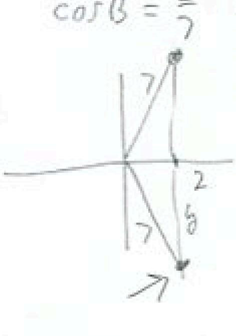
$4 + y^2 = 25$

$y^2 = 21$

$y = \pm\sqrt{21}$

$= -\sqrt{21}$

$\cos\beta = \frac{2}{7}$



$2^2 + y^2 = 7^2$

$y^2 = 45$

$y = \pm\sqrt{45} = -3\sqrt{5}$

$\sin\beta = -\frac{3\sqrt{5}}{7}$