**Trigonometry — Lecture notes**
**MAT 145, Spring 2025 — D. Ivanšić**

# 8.1 Law of Sines

At the beginning of the course, we *solved* right triangles. This meant, given two independent pieces of information about a right triangle (a side and an angle, or two sides) we found all the sides and all the angles. (Note that the two acute angles in a right triangle are not independent due to $\beta = 90° - \alpha$.)

Now we wish to solve *oblique* triangles, the triangles that are not right.

| | all angles acute | one angle obtuse ($> 90°$) |
|---|---|---|
| Two shapes of oblique triangles are possible: | | |

Note a triangle cannot have more than one obtuse angle, since otherwise the sum of all angles exceeds 180°.

We adopt the usual labeling standard, where angles $A, B, C$ are opposite sides $a, b, c$, respectively.

For an oblique triangle, three independent pieces of information about the triangle are needed. (For a right triangle, it is really the same, it's just that the third piece of information is that one angle is 90°.) The following three pieces of information determine a triangle.

| ASA | AAS | SSA |
|---|---|---|
| side and two angles adjacent to it | two angles and side opposite one of them | two sides and angle opposite one side |

| SAS | SSS |
|---|---|
| two sides and angle between them | three sides |

Our first tool in solving triangles is:

**Law of sines.** In a triangle with angles $A, B, C$ opposite sides $a, b, c$ we have:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

*Proof.*

**Example.** Solve the ASA triangle where $c = 5$, $A = 35°$, $B = 70°$. First explain why there is exactly one solution.

**Example.** Solve the AAS triangle where $a = 7$, $A = 45°$, $B = 40°$. First explain why there is exactly one solution.

**Example.** Solving an SSA triangle involves several possibilities:

1) Side opposite the angle is too short: no solution.

2) Side opposite the angle is just long enough to reach the base: one solution or none

3) Side opposite the angle is smaller than the adjacent side and reaches base in two places: two solutions or none.

4) Side opposite the angle is shorter than the adjacent side: one solution.

**Example.** Solve the SSA triangle where $A = 40°$, $a = 2$, $b = 4$.

**Example.** Solve the SSA triangle where $C = 50°$, $a = 5$, $c = 4$.

**Example.** Solve the SSA triangle where $B = 60°$, $b = 7$, $c = 5$.

**Area of a triangle.** The area of a triangle equals half of product of lengths of two sides and the sine of the angle between them. (Note that all the variables are different letters.)

$$\text{Area} = \frac{1}{2}ab\sin C$$

*Proof.*

**Example.** Find the area of the triangle where $c = 8\,\text{in}$, $B = 38°$, $C = 123°$.

## 8.2 Law of Cosines

Note that the Law of Sines does not help with SAS or SSS triangles, buecause we get equations with more than one unknown.

**Law of Cosines.** In a triangle with angles $A, B, C$ opposite sides $a, b, c$ we have:

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad b^2 = a^2 + c^2 - 2ac\cos B \qquad a^2 = b^2 + c^2 - 2bc\cos A$$

*Proof.*

**Example.** Solve the SAS triangle where $a = 3$, $b = 4$, $C = 55°$. First explain why there is exactly one solution.

**Example.** Solve the SSS triangle where $a = 3$, $b = 7$, $c = 5$. First explain why there is exactly one solution, as long as the sum of lengths of every pair of sides is bigger than the third one.

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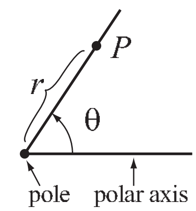
# 8.4 Polar Coordinates

To specify the position of a point in the plane, we have so far used rectangular (Cartesian) coordinates.

Fix a ray, called the *polar axis*, along which we will place the initial side of an angle, usually the positive $x$-axis. We may also specify the position of a point by giving

1) the angle $\theta$ on whose terminal side point is located.
2) the distance $r$ from the origin.

We say $(r, \theta)$ are *polar coordinates* of the point $P$.




**Example.** Sketch the points with polar coordinates:

$\left(5, \frac{\pi}{2}\right)$ $\left(3, -\frac{5\pi}{6}\right)$ $(4, 180°)$

In polar coordinates, we allow $r$ to be negative (so, $r$ is "directed distance"). If $r < 0$ we go distance $|r|$ from the origin along the ray that is opposite to terminal side of angle $\theta$.

**Example.** Sketch the points with polar coordinates:

$\left(-2, \frac{\pi}{2}\right)$ $(-3, \pi)$ $(-4, 30°)$

**Example.** In contrast to rectangular coordinates, every point has many polar coordinates. List all the possible coordinates of the point with polar coordinates $\left(4, \frac{\pi}{3}\right)$

**Converting coordinates.** Using formulas $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$ we get the conversion formulas at right.

| polar → rectangular | rectangular → polar |
| --- | --- |
| $x = r\cos\theta$ | $r = \sqrt{x^2 + y^2}$ |
| $y = r\sin\theta$ | $\tan\theta = \dfrac{y}{x}$ |

**Example.** Convert to rectangular coordinates.

$\left(5, \dfrac{5\pi}{6}\right)$ $(-3, 60°)$

**Example.** Convert to polar coordinates.

$(2, 2)$ $(-3, \sqrt{3})$ $(-3, -7)$

Convert the following equations to polar coordinates.

**Example.** $x^2 + y^2 = 9$

**Example.** $y = 3x + 5$

Convert the following equations to rectangular coordinates.

**Example.** $r = \sin\theta$

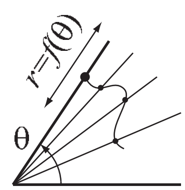
**Example.** $r = \sin(2\theta)$

Curves with basic equations in $r, \theta$ can be drawn fairly easily.

**Example.** Sketch the curves given by the following equations.

$r = 2$ $\qquad\qquad\qquad$ $\theta = \dfrac{\pi}{3}$

A polar equation typically has the form $r = f(\theta)$, so $r$ varies as a function of $\theta$. We can imagine the curve as the path of a bead moving back an forth along the terminal side of the angle as it circles around the origin.




Graph the following polar equations.

**Example.** $r = 1 + \sin\theta$, $\theta$ in $[0, 2\pi]$

**Example.** $r = \cos(2\theta)$, $\theta$ in $[0, 2\pi]$

**Example.** $r = 4\sin(3\theta)$, $\theta$ in $[0, 2\pi]$

$$\left.\begin{array}{l} r = a\sin(n\theta) \\ r = a\cos(n\theta) \end{array}\right\} \text{ is a rose with } \left\{\begin{array}{l} n \text{ petals, if } n \text{ is odd (traversed once for } 0 \le \theta \le \pi) \\ 2n \text{ petals, if } n \text{ is even (traversed once for } 0 \le \theta \le 2\pi) \end{array}\right.$$

**Example.** $r = \theta$, $\theta$ in $[0, \infty]$

**Example.** $r = f(\theta)$, where $f$ is given by the graph below.