Trigonometry — Lecture notes
MAT 145, Spring 2025 — D. Ivanšić

7.1 Identities: Pythagorean and Sum and Difference

Identities involving trigonometric functions that we have had so far:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sin(-\theta) = -\sin\theta$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \cos(-\theta) = \cos\theta$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan(-\theta) = -\tan\theta$$

Pythagorean identities. Come from the equation of the unit circle: $x^2 + y^2 = 1$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Example. Use established identities to simplify expression.

$(1 - cos^2\theta)(1 + \cot^2\theta) =$

Example. Use established identities to simplify expression.

$\dfrac{\cos^2\theta}{1 + \sin\theta} =$

Example. Use the distance formula between points (x_1, y_1) and (x_2, y_2),

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

to find a formula for $\cos(u - v)$.

Sum and difference formulas.

$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

$\tan(u \pm v) = \dfrac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$

Example. Now we can find exact values of trigonometric functions of some other angles.

$\sin 15° =$

$\cos \dfrac{11\pi}{12} =$

Example. If $\sin u = \frac{1}{3}$ and u is in the second quadrant and $\cos v = -\frac{2}{5}$ and v is in the third quadrant, find $\sin(u+v)$.

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7.2 Identities: Cofunction, Double-Angle and Half-Angle

Recall the cofunction identities: cofunctions of complementary angles are equal.

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ $\quad\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$ $\quad\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$

Example.

$\sin\left(\frac{3\pi}{5}\right)=$ $\quad\csc\left(\frac{4\pi}{7}\right)=$

Double-Angle Formulas.

$\sin(2\theta)=2\sin\theta\cos\theta$ $\quad\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$\cos(2\theta)=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$

Example. If $\csc\theta=4$ and θ is in the second quadrant, find $\sin(2\theta)$ and $\cos(2\theta)$. In what quadrant is 2θ?

Half-Angle Formulas.

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \qquad \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \qquad \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$$

Example. Determine $\sin\frac{\pi}{12}$, $\cos\frac{\pi}{12}$.

Example. Simplify the expression.

$(\sin\theta+\cos\theta)^2 =$

7.3 Proving Trigonometric Identities

Using known identities, we can prove new ones. To prove an identity start with one side and try to transform it so you get the other. Usually we start with the more complicated side, because it is typically easier to think of ways to simplify an expression (after all, that is what we are trained to do in math classes) than of ways to write a simple expression in a more complicated way.

Example. Show that $\sec\theta\sin\theta = \tan\theta$.

Example. Show that $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta$.

Example. Show that $\sec(2\theta) = \dfrac{\sec^2\theta}{2 - \sec^2\theta}$.

7.4 Inverses of Trigonometric Functions

Example. Recall that we solved equations such as $\sin\theta = 0.46$ by applying $\sin^{-1}$ on the calculator:

$$\sin\theta = 0.46 \Longrightarrow \theta = \sin^{-1} 0.46 = 27.387108^\circ$$

This was straightforward for acute angles, but has twists for general angles.

Note. From now on, arcsin, arccos and arctan will be used for $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ to reduce the chances of confusing inverse and reciprocal.

Example. Determine $\arcsin\frac{1}{2}$ and $\arcsin\left(-\frac{1}{2}\right)$, that is, angles whose sine is $\frac{1}{2}$ and $-\frac{1}{2}$.

We set things up so that arcsin is exactly one number (so arcsin is a function). We choose:

Definition. $\arcsin x$ is the unique angle whose sine is x and is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore: $\arcsin\frac{1}{2} =$ $\qquad\qquad$ $\arcsin\left(-\frac{1}{2}\right) =$

Example. Determine $\arccos\frac{1}{2}$ and $\arccos\left(-\frac{1}{2}\right)$, that is, angles whose cosine is $\frac{1}{2}$ and $-\frac{1}{2}$.

We set things up so that arccos is exactly one number (so arccos is a function). We choose:

Definition. $\arccos x$ is the unique angle whose cosine is x and is in the interval $[0, \pi]$.

Therefore: $\arccos\frac{1}{2} =$ $\qquad$ $\arccos\left(-\frac{1}{2}\right) =$

Example. Determine the following values of inverse trigonometric functions:

$\arcsin 1 =$ $\qquad$ $\arcsin\left(-\frac{\sqrt{3}}{2}\right) =$ $\qquad$ $\arccos 0 =$ $\qquad$ $\arccos\left(-\frac{\sqrt{2}}{2}\right) =$

Example. What is $\arcsin 1.25$?

Example. Determine $\arctan \dfrac{1}{\sqrt{3}}$ and $\arcsin\left(-\dfrac{1}{\sqrt{3}}\right)$, that is, angles whose tangent is $\dfrac{1}{\sqrt{3}}$ and $-\dfrac{1}{\sqrt{3}}$.

We set things up so that arctan is exactly one number (so arctan is a function). We choose:

Definition. $\arctan x$ is the unique angle whose tangent is x and is in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Therefore: $\arctan \dfrac{1}{\sqrt{3}} =$ $\qquad$ $\arctan\left(-\dfrac{1}{\sqrt{3}}\right) =$

Note. $\dfrac{\pi}{2}$ cannot be the value of $\arctan x$ because $\tan \dfrac{\pi}{2}$ is not defined.

Domains and ranges of standard inverse trigonometric functions.

Domain of arcsin: $[-1, 1]$	Range of arcsin: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Domain of arccos: $[-1, 1]$	Range of arccos: $[0, \pi]$
Domain of arctan: $(-\infty, \infty)$	Range of arctan: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example. Determine the following values of inverse trigonometric functions:

$\arctan 1 =$ $\qquad$ $\arctan \sqrt{3} =$ $\qquad$ $\arctan 0 =$ $\qquad$ $\arctan(-1) =$

Important subtlety.

$\sin(\arcsin x) = x$	$\arcsin(\sin \theta)$ is not always θ, but is if θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos(\arccos x) = x$	$\arccos(\cos \theta)$ is not always θ, but is if θ is in $[0, \pi]$
$\tan(\arctan x) = x$	$\arctan(\tan \theta)$ is not always θ, but is if θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
The sine of the angle whose sine is x is ... x (duh!)	The angle whose sine equals $\sin \theta$ is not necessarily equal to θ, because many angles have the same sine.

Example. Simplify using a picture.

$\arcsin\left(\sin \frac{3\pi}{2}\right) =$ $\qquad$ $\arccos\left(\cos \frac{4\pi}{3}\right) =$

Example. Simplify using a picture.

$\arctan\left(\tan\left(-\frac{5\pi}{8}\right)\right) =$ $\qquad$ $\arcsin\left(\sin \frac{3\pi}{5}\right) =$ $\qquad$ $\arccos\left(\cos\left(-\frac{8\pi}{7}\right)\right) =$

Example. Simplify.

$\tan\left(\arcsin\left(-\frac{2}{7}\right)\right) =$

Example. Simplify.

$\sec(\arctan u) =$

Often used graph.

$y = \arctan x$

7.5 Solving Trigonometric Equations

Example. Solve the equation in general, and then in $[0, 2\pi)$.

$2\sin\theta = \sqrt{2}$

Note. The solution is not just $\arcsin\frac{\sqrt{2}}{2}$ because we ask for *all* angles whose sine is $\frac{\sqrt{2}}{2}$, not just the one in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that $\arcsin\frac{\sqrt{2}}{2}$ provides.

Example. Solve the equation in general, and then in $[0, 2\pi)$.

$2\cos\theta + 1 = 0$

Example. Solve the equation in general, and then in $[0, 2\pi)$.

$3\sin\theta - 7 = 4$

Example. Solve the equation in general, and then in $[0, 2\pi)$.

$5 \sin \theta - 1 = 2$

Example. Solve the equation in general, and then in $[0, 2\pi)$.

$2 \cos^2 \theta + \cos \theta - 1 = 0$

Example. Solve the equation in general, and then in $[0, 2\pi)$.

$1 - \sin \theta = 2 \cos^2 \theta$

Example. Solve the equation in general, and then in $[0, 2\pi)$.

$\sec^2 \theta + \tan \theta = 19 - 2 \tan \theta$

Example. An equation that has a mix of trigonometric and nontrigonometric functions is typically impossible to solve algebraically, but we can use a graphing calculator to find approximate solutions. Solve.

$x^2 + 7 \cos x = x$