Trigonometry — Lecture notes
MAT 145, Spring 2025 — D. Ivanšić

6.1 Trigonometric Functions of Acute Angles

An acute angle is an angle whose measure is between 0° and 90°.

Recall a few facts about triangles and right triangles.

In any triangle, the sum of angles is 180°.

$$\alpha + \beta + \gamma = 180^\circ$$

If one of the angles is 90°, the triangle is called a *right* triangle. In that case, the sum of remaining angles is 90°.

$$\alpha + \beta = 90^\circ$$

The sides in a right triangle satisfy the *Pythagorean Theorem*: sum of squares of the shorter sides equals the square of the longest side.

$$a^2 + b^2 = c^2$$

Triangles are *similar* if they have the same shape (angles), but possibly different sizes. When triangles are similar, the ratios of the corresponding sides are equal.

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

If two right triangles have one acute angle the same, then the other acute angle is the same, too, so all angles are same and the triangles are similar.

$$\frac{a}{b} = \frac{a'}{b'} \quad \frac{a}{c} = \frac{a'}{c'} \quad \frac{b}{c} = \frac{b'}{c'}$$

Definition: If θ is an acute angle, take any right triangle that has one acute angle equal to θ. Then define the trigonometric functions sine, cosine, tangent, secant, cosecant and cotangent as follows. The abbreviations hyp, opp and adj stand for "hypothenuse," "opposite" and "adjacent" (of the angle θ in a right triangle).

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\tan\theta}$$

Notice the left column can be memorized with the mnemonic SOHCAHTOA. Also notice that any two right triangles with the same acute angle give you the same ratios because the triangles are similar.

Example: Find the values of trigonometric functions for an angle θ that is adjacent to a side of length 7 in a right triangle, where the other side has length 5.

Example: Find the values of trigonometric functions for angles 30°, 45° and 60°. These are called the *standard angles.*

Most important is to memorize possible values and their relative sizes:
for sines and cosines: $\frac{1}{2} < \frac{\sqrt{2}}{2} < \frac{\sqrt{3}}{2}$ for tangents: $\frac{1}{\sqrt{3}} < 1 < \sqrt{3}$

If we know the value of one trigonometric function of an angle, we can find the other ones.

Example: Suppose $\cos\theta = \frac{1}{3}$. Find all the other trigonometric function values of θ.

Definition: Angles are *complementary* if the sum of their measures is 90°. The *complement of an angle* α is an angle with measure $90° - \alpha$.

In a right triangle, the acute angles are complementary and the complement of either acute angle is the other one.

$\alpha + \beta = 90°$, so $\beta = 90° - \alpha$

Definition: Trigonometric functions are paired as follows: sin-cos, tan-cot, sec-csc (the name of the second function is the name of the first one with the prefix "co-"). The *cofunction* of a trigonometric function is the other function in the pair. For example, the cofunction of sine is cosine, and of cosine is sine.

Cofunction Identities. Cofunction values of complementary angles are equal.

$\sin\theta = \cos(90° - \theta)$	$\tan\theta = \cot(90° - \theta)$	$\sec\theta = \csc(90° - \theta)$
$\cos\theta = \sin(90° - \theta)$	$\cot\theta = \tan(90° - \theta)$	$\csc\theta = \tan(90° - \theta)$

Example: $\sin 12° = \cos$ ____ $\qquad$ $\csc 35° = \sec$ ____ $\qquad$ $\cot 74° = \tan$ ____

Example. Evaluate the trigonometric function values on the calculator to six decimal places. Make sure you are in degree mode.

$\sin 50^\circ =$ $\quad\quad$ $\sec 26^\circ =$ $\quad\quad$ $\cot 75^\circ =$

Example. Find the acute angle whose cosine is equal to 0.4.

Further subdivisions of degrees: minutes, seconds.

$1^\circ = 60'$ (minutes) $= 3600''$ (seconds) $\quad\quad$ $1'$ (minute) $= 60''$ (seconds)

6.2 Applications of Right Triangles

Example. A 20 foot ladder leans against the wall and forms a 23° angle with the wall.
a) How far up the wall does the top of the ladder touch the wall?
b) How far is the bottom of the ladder from the wall?

Example. A 250-foot tall cell-phone tower is secured by three 175-foot guy wires that are anchored 125 feet up the tower.
a) What angle do the wires form with the ground?
b) How far from the base of the tower are the guy wires anchored?

Example. Standing away from a building, you measure the angle of elevation to the top of the building to be $34°$. Then you move 50 meters towards the building and find the angle of elevation to now be $62°$.
a) How tall is the building?
b) How far from the building were you initially standing?

Methods of attack of these problems:
1) Familiarize yourself with the problem: read it carefully (more than once, if needed) What is given, and what are you asked to find: assign a variable (x, t, u, etc.) to represent what you are asked to find. Draw the @!♣#♢$ picture! Large, clear, with indicated quantities.
2) From the picture, write a trigonometric equation that represents the facts of the problem. Use the Pythagorean theorem, SOHCAHTOA, or another theorem that we will yet learn.
3) Solve the equation.
4) Reality check: does your answer make sense? Check with facts of problem.

6.3 Trigonometric Functions of Any Angle

Definitions. A point on a line divides the line into two infinite pieces called *rays*. (One could say a ray is half of a line.) A ray has an endpoint.

An *angle* is the union of two rays with a common endpoint.

In trigonometry, we typically think of an *angle as a rotation* that moves one side of the angle (called *initial side*) to the other side of the angle (called *terminal side*).

The initial side is usually drawn along the positive x-axis. Rotation is called *positive* if it is in the counterclockwise direction, *negative* if it is in the clockwise direction.

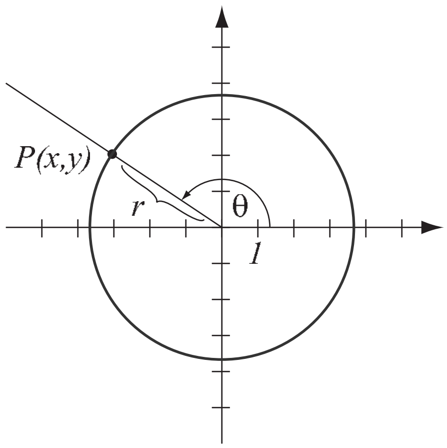
If two angles have the same terminal side, we call them *coterminal*. Note that measures of coterminal angles differ by a multiple of 360°.

$410°, 50°$ $\quad$ $125°, -235°$ $\quad$ $-90°, 630°$

Angles are *complementary* if their measures add up to 90° and *supplementary* if their measures add up to 180°.

$15° + 75° = 90°$ $\quad$ $115° + 65° = 180°$

Definition. To define the trigonometric function of any angle, position the angle so its initial side is along the positive x-axis and use any point P on the terminal side of the angle.



$$r = \sqrt{x^2 + y^2} \quad \text{(distance from } P \text{ to the origin)}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x} = \frac{1}{\cos\theta}$$

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y} = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y} = \frac{1}{\tan\theta}$$

Note. For reasons like we had for similar triangles, all the ratios are the same regardless of which point P on the terminal side of the angle are chosen.

Note. For acute angles, the definition agrees with SOHCAHTOA.

Example. Find the values of trigonometric functions for an angle whose terminal side goes through point $(1, -3)$.

Example. Find the values of trigonometric functions of $270°$.

Example. If $\csc\theta = -\dfrac{3}{2}$ and θ is in the 2nd quadrant, find the other trigonometric functions of θ.

6.4 Radians, Arc Length and Angular Speed

Definition. The *unit circle* is the circle centered at the origin with radius 1.

Its equation is $x^2 + y^2 = 1$.

We may measure rotation (and recall that we consider angles as rotations) by measuring the distance a point has traveled along the unit circle starting from point $(1, 0)$ to a point on the terminal side.

If a point has traveled s units, we say the corresponding angle has measure s radians.

Example. What are the measures, in radians, of these angles?

Note. 1 radian$\approx 57.3°$

Conversion radians $\longleftrightarrow$ degrees.

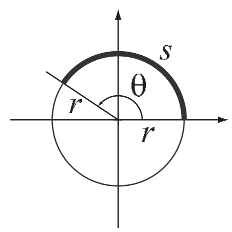
$$\pi \text{ radians } = 180° \qquad 1 \text{ radian } = \frac{180°}{\pi} \qquad 1° = \frac{\pi}{180} \text{ radians}$$

Example. Convert 75° to radians.

Example. Convert 4.5 radians to degrees.

Note. Meaures of coterminal angles in radians differ by a multiple of 2π.

Arc length. Suppose angle of θ radians subtends an arc of length s on a circle of radius r. We find the relationship between s, r and θ.



$$\theta = \frac{s}{r}$$

$$s = r\theta$$

θ measured in radians

Example. Find the length of an arc on a circle of radius 7 cm subtended by an angle of $\frac{3\pi}{2}$ radians.

Recall the formula: distance = rate · time which gives $\text{rate} = \dfrac{\text{distance}}{\text{time}}$

$$s = v \cdot t \qquad v = \frac{s}{t}$$

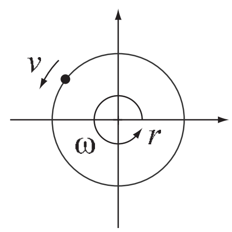
Similarly, for a rotating object we can define *angular speed* ω:

$$\text{angular speed} = \frac{\text{angle traversed}}{\text{time elapsed}} \qquad \omega = \frac{\theta}{t}$$

Example. Determine the angular speed of an engine running at 1500rpm.

Example. Determine the angular speed of the minute hand on a clock.

Connection between angular and linear speed.



$v = r\omega$

ω measured in radians per time unit

Example. What is the linear speed of Murray, KY, as it rotates around the Earth's axis of rotation on a circle of radius 3178 mi?

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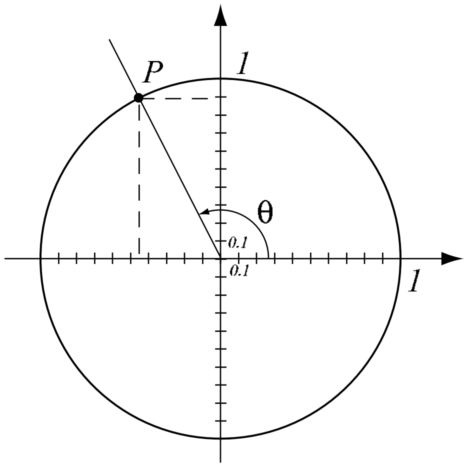
6.5 Trigonometric Functions Using the Unit Circle



When point P on terminal side of angle θ is on the unit circle, the following hold:

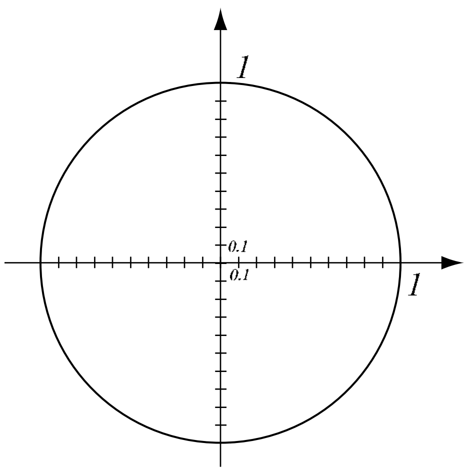
$$\cos\theta = x\text{-coordinate of } P \qquad \sec\theta = \frac{1}{x}$$

$$\sin\theta = y\text{-coordinate of } P \qquad \csc\theta = \frac{1}{y}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

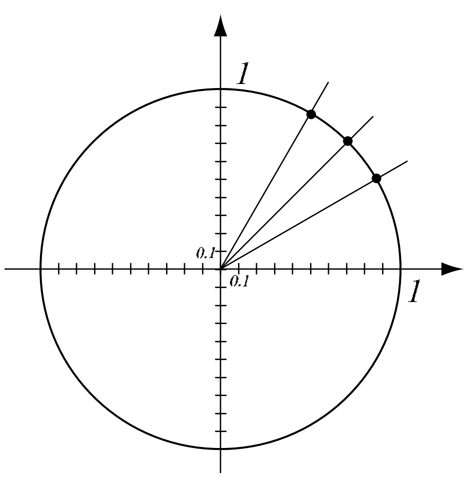
In the example: $\cos\theta \approx -0.45$, $\sin\theta \approx .89$, $\tan\theta \approx 0.89/(-0.45) \approx -1.98$

Example. Use the picture below to estimate the given trigonometric functions. Compare your answer with results you get with a calculator.



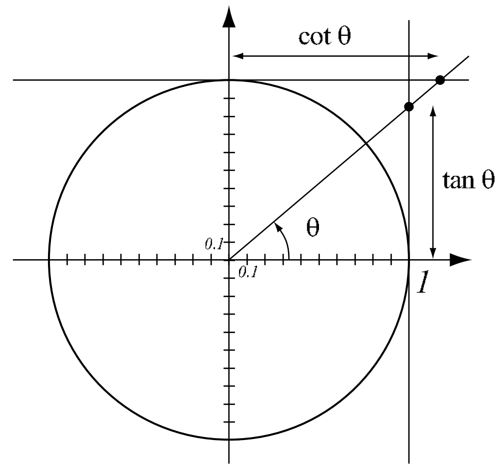
	estimate	calculator
$\cos 100^\circ =$		
$\sin 100^\circ =$		
$\cos 250^\circ =$		
$\sin 250^\circ =$		
$\cos\left(-\frac{3\pi}{8}\right) =$		
$\sin\left(-\frac{3\pi}{8}\right) =$		

Example. Indicate the exact coordinates of the points P corresponding to angles of 30°, 45° and 60°. Choose among $\frac{1}{2} < \frac{\sqrt{2}}{2} < \frac{\sqrt{3}}{2}$ by comparing the sizes of the x- and y-coordinates in the picture. Then state the values of the trigonometric functions at right.

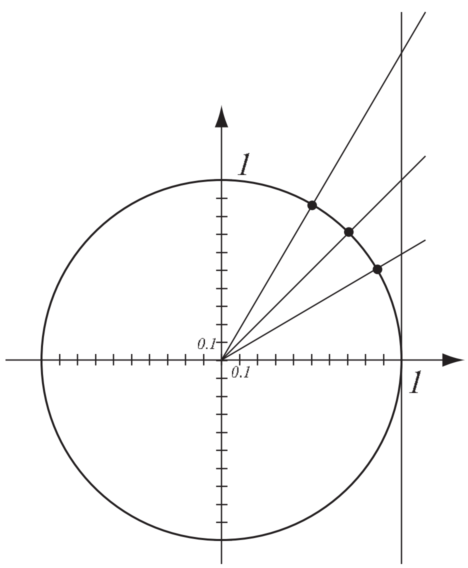


	exact value
$\cos 30^\circ = \cos\frac{\pi}{6} =$	
$\sin 30^\circ = \sin\frac{\pi}{6} =$	
$\cos 45^\circ = \cos\frac{\pi}{4} =$	
$\sin 45^\circ = \sin\frac{\pi}{4} =$	
$\cos 60^\circ = \cos\frac{\pi}{3} =$	
$\sin 60^\circ = \sin\frac{\pi}{3} =$	

Example. The values of $\tan\theta$ and $\cot\theta$ can be read from the picture at right as shown.



Then use the picture below to find the tangent and cotangent of angles 30°, 45° and 60°. Choose among $\frac{1}{\sqrt{3}} < 1 < \sqrt{3}$ by comparing the sizes of the cuts on the reference line for tangent.



$$\tan 30^\circ = \tan\frac{\pi}{6} =$$

$$\cot 30^\circ = \cot\frac{\pi}{6} =$$

$$\tan 45^\circ = \tan\frac{\pi}{4} =$$

$$\cot 45^\circ = \cot\frac{\pi}{4} =$$

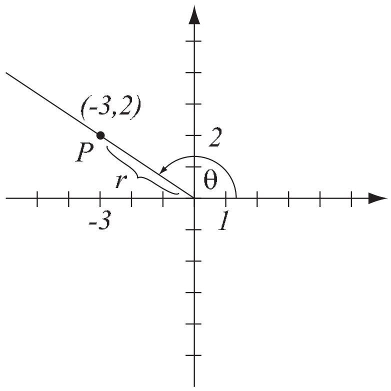
$$\tan 60^\circ = \tan\frac{\pi}{3} =$$

$$\cot 60^\circ = \cot\frac{\pi}{3} =$$

Example. Draw the unit circle and the appropriate angle in order to infer from the picture the exact values of the following trigonometric expressions.

$\sin 120^\circ =$ $\quad$ $\tan\frac{5\pi}{3} =$ $\quad$ $\csc(-135^\circ) =$ $\quad$ $\cos\frac{3\pi}{2} =$

$\cot(-150^\circ) =$ $\quad$ $\sec\left(-\frac{7\pi}{6}\right) =$ $\quad$ $\sin 450^\circ =$ $\quad$ $\tan\left(-\frac{11\pi}{3}\right) =$



Recall that when point P on terminal side of angle θ is *not* on the unit circle, then slightly different formulas apply:

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{13}}$$

$$\sin\theta = \frac{y}{r} = \frac{2}{\sqrt{13}}$$

$$\tan\theta = \frac{y}{x} = -\frac{2}{3}$$

Graph of $\sin\theta$. Use the unit circle to graph the function $y = \sin\theta$.

Domain: Periodic with period 2π: $\sin(\theta + 2\pi) = \sin\theta$

Range: Odd function: $\sin(-\theta) = -\sin\theta$

Graph of $\cos\theta$. Use the unit circle to graph the function $y = \cos\theta$.

Domain: Periodic with period 2π: $\cos(\theta + 2\pi) = \cos\theta$

Range: Even function: $\cos(-\theta) = \cos\theta$

Graph of $\tan\theta$. Use the unit circle to graph the function $y = \tan\theta$.

Domain:

Range:

Periodic with period π: $\tan(\theta+\pi) = \tan\theta$

Odd function: $\tan(-\theta) = -\tan\theta$

Vertical asymptotes: $x = (2k+1)\frac{\pi}{2}$

Graph of $\cot\theta$. Use the unit circle to graph the function $y = \cot\theta$.

Domain:

Range:

Periodic with period π: $\cot(\theta+\pi) = \cot\theta$

Odd function: $\cot(-\theta) = -\cot\theta$

Vertical asymptotes: $x = k\pi$

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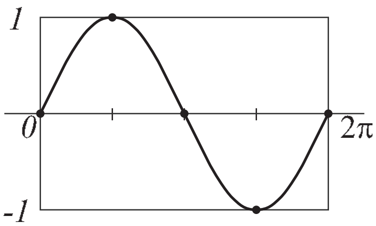
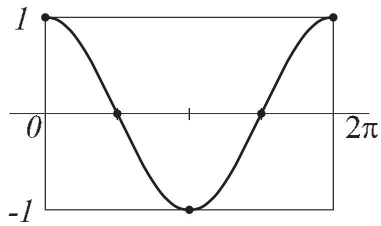
6.6 Graphs of Transformed Sine and Cosine Functions

We wish to graph $y = A\sin(Bx + C) + D$ or $y = A\cos(Bx + C) + D$.

Recall that a basic period of $\sin x$ and $\cos x$ fit in a box with five characteristic points:



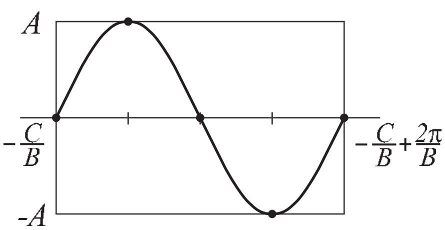
$y = \sin x$



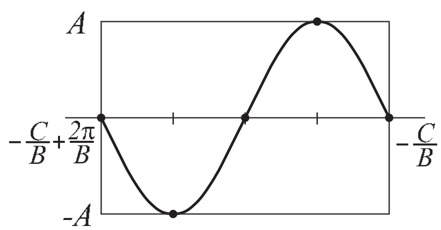
$y = \cos x$

The graph of $y = A\sin(Bx + C) = A\sin\left(B\left(x + \frac{C}{B}\right)\right)$ is obtained from $y = \sin x$ by stretches and shifts.

The result is a graph that fits in a modified box with five characteristic points. Period is $\frac{2\pi}{B}$. A is called the *amplitude* of the graph, telling us how far above and below the centerline the graph goes.



$y = A\sin(Bx+C)$, $B>0$



$y = A\sin(Bx+C)$, $B<0$

$A>0$
both graphs

Example. Sketch the graph of $y = 3\sin\left(2x - \frac{\pi}{2}\right)$. Draw two periods.

To sketch the graph of $y = A\sin(Bx + C) + D$, first do $y = A\sin(Bx + C)$, and then shift it up by D.

Example. Sketch the graph of $y = 4\cos(-3x + \pi) + 2$. Draw two periods.