Calculus 1 — Final Exam
MAT 250, Fall 2025 — D. Ivanšić

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Show all your work!

1. (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x\to -3+} f(x) = -\infty$

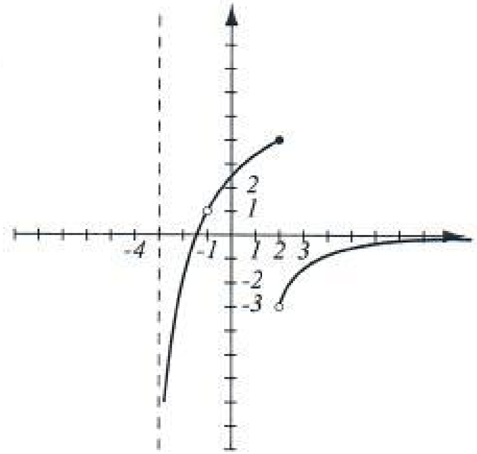
$\lim_{x\to 2-} f(x) = 4$

$\lim_{x\to 2+} f(x) = -3$

$\lim_{x\to 2} f(x) =$ DNE because one-sided limits are not same

$\lim_{x\to -1} f(x) = 1$

$\lim_{x\to \infty} f(x) = 0$



List points where f is not continuous and explain why.

f is not continuous at $x=1$ – $f(1)$ is not defined

$x=2$ – $\lim_{x\to 2} f(x)$ DNE

Find the following limits algebraically. Do not use L'Hospital's rule.

2. (6pts) $\lim_{x\to 2+} \dfrac{x^2-3}{6-3x} = \dfrac{1}{0-} = -\infty$

$6-3x$

$+$ $-$

2

3. (6pts) $\lim_{x\to 3} \dfrac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3} \dfrac{x-3}{\sqrt{x}-\sqrt{3}} \cdot \dfrac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} = \lim_{x\to 3} \dfrac{\cancel{(x-3)}(\sqrt{x}+\sqrt{3})}{\cancel{x-3}}$

$= \sqrt{3}+\sqrt{3} = 2\sqrt{3}$

4. (8pts) Find $\lim_{x\to 0} x^2\left(2\cos\frac{3}{x}+1\right)$. Use the theorem that rhymes with what students pay to a university in addition to tuition.

$$-1 \le \cos\frac{3}{x} \le 1$$

$$-2 \le 2\cos\frac{3}{x} \le 2$$

$$-1 \le 2\cos\frac{3}{x}+1 \le 3$$

$$-x^2 \le x^2\left(2\cos\frac{3}{x}+1\right) \le 3x^2$$

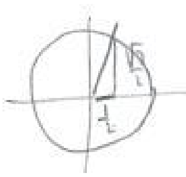
$$\left.\begin{array}{l}\lim_{x\to 0}(-x^2)=0\\ \lim_{x\to 0}3x^2=0\end{array}\right\}$$ equal, so by squeeze theorem

$$\lim_{x\to 0} x^2\left(2\cos\frac{3}{x}+1\right)=0$$

5. (10pts) Write the equation of the tangent line to the curve $y=\sin^3\theta$ at point $\theta=\frac{\pi}{3}$.

$$y' = 3\sin^2\theta\cos\theta$$

$$y'\left(\frac{\pi}{3}\right) = 3\sin^2\frac{\pi}{3}\cos\frac{\pi}{3} = 3\cdot\left(\frac{\sqrt{3}}{2}\right)^2\cdot\left(\frac{1}{2}\right) = 3\cdot\frac{3}{4}\cdot\frac{1}{2} = \frac{9}{8}$$



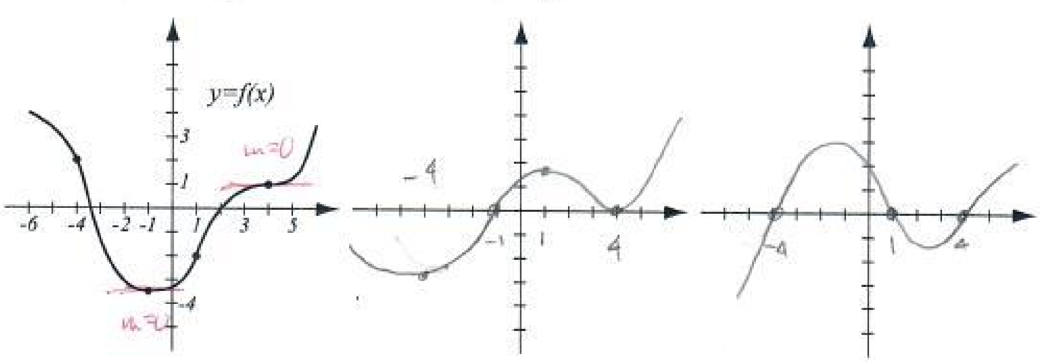
Tangent line $y-\frac{3\sqrt{3}}{8}=\frac{9}{8}\left(x-\frac{\pi}{3}\right)$

$$y\left(\frac{\pi}{3}\right) = \left(\sin\frac{\pi}{3}\right)^3 = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$$

$$y = \frac{9}{8}x - \frac{3\pi}{8} + \frac{3\sqrt{3}}{8}$$

$$y = \frac{9}{8}x + \frac{3\sqrt{3}-3\pi}{8}$$

6. (10pts) The graph of f is given. Use it to draw the graphs of f' and f'' in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of f. The relevant special points have been highlighted.



7. (26pts) Let $f(x) = \dfrac{x}{e^x}$. The domain of this function is all real numbers (you do not have to verify this). Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

d) Use information from a)–c) to sketch the graph.

$$f'(x) = \frac{1\cdot e^x - x\cdot e^x}{(e^x)^2} = \frac{e^x(1-x)}{(e^x)^2} = \frac{1-x}{e^x}$$

$$f''(x) = \frac{(-1)e^x - (1-x)e^x}{(e^x)^2} = \frac{e^x(-1-1+x)}{(e^x)^2} = \frac{x-2}{e^x}$$

$e^x > 0$ for all x

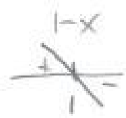
so f', f'' have sign of numerator

a) $f'(x)$ always defined

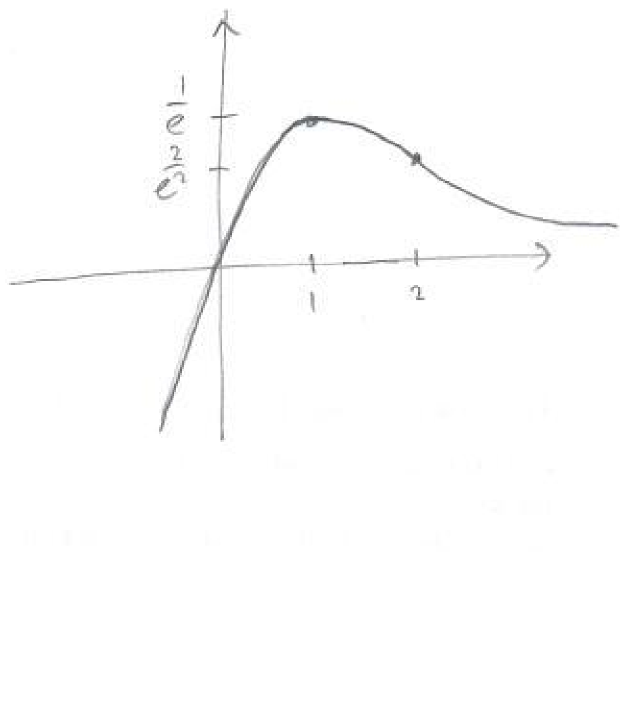
$f'(x) = 0$ when $1 - x = 0$

$x = 1$

	$x<1$	1	$x>1$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	loc. max	$\searrow$



x	$\frac{x}{e^x}$
1	$\frac{1}{e} \approx \frac{1}{3}$
2	$\frac{2}{e^2} \approx \frac{2}{9}$



b) $f''(x)$ always defined

	$x<2$	2	$x>2$
$f''(x)$	$-$	0	$+$
$f(x)$	CD	IP	CU

c) $$\lim_{x\to\infty} \frac{x}{e^x} = \left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

$$\lim_{x\to-\infty} \frac{x}{e^x} = \frac{-\infty}{0+} = -\infty \cdot \frac{1}{0+} = -\infty\cdot\infty = -\infty$$

8. (8pts) Let $f(x) = 2x - x^2$. Find the absolute minimum and maximum values of f on the interval $[0, 4]$. $f'(x) = 2-2x$

$f'(x)$ is always defined

$f'(x)=0$

$2-2x=0$

$2x=2$

$x=1$

x	$2x-x^2$	
0	0	
4	$8-16=-8$	abs min
1	$2-1=1$	abs max

9. (8pts) Consider the integral $\int_0^4 2x - x^2\,dx$.

a) Use the inequality $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$, where $m \le f(x) \le M$ on $[a, b]$, to give an estimate of the integral. (Information from previous problem gives you m and M.)

b) Evaluate the integral and verify your estimate from a).

a) On $[0,4]$

$-8 \le 2x-x^2 \le 1$

$-8(4-0) \le \int_0^4 2x-x^2\,dx \le 1\cdot(4-0)$

$-32 \le \int_0^4 2x-x^2\,dx \le 4$

b) $\int_0^4 2x-x^2\,dx = x^2-\frac{x^3}{3}\Big|_0^4$

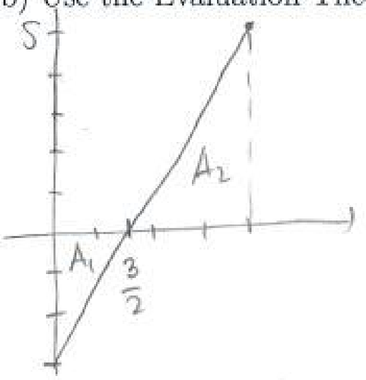
$= 4^2-\frac{4^3}{3} = 16-\frac{64}{3}-0 = \frac{48-64}{3} = -\frac{16}{3}$

$-32 \le -\frac{16}{3} \le 4$ is true

10. (10pts) Consider the integral $\int_0^4 2x - 3\,dx$.

a) Use a picture and the "area" interpretation of the integral to determine the value of the integral.

b) Use the Evaluation Theorem to find the integral and verify your conclusion from a).



a) $\int_0^4 2x-3\,dx = -A_1 + A_2 = -\frac{1}{2}3\cdot\frac{3}{2} + \frac{1}{2}\frac{5}{2}\cdot 5$

$= -\frac{9}{4}+\frac{25}{4} = \frac{16}{4} = 4$

b) $\int_0^4 2x-3\,dx = (x^2-3x)\Big|_0^4 = 4^2-3\cdot 4-0 = 16-12 = 4$

11. (6pts) Use implicit differentiation to find y' in general:

$x^2y^2 = x + y \quad \frac{d}{dx}$

$2xy^2 + x^2 2yy' = 1 + y'$

$2x^2yy' - y' = 1 - 2xy^2$

$y'(2x^2y - 1) = 1 - 2xy^2$

$y' = \dfrac{1 - 2xy^2}{2x^2y - 1}$

12. (6pts) Find $f(x)$ if $f'(x) = \sqrt[5]{x} - \dfrac{1}{x}$ and $f(1) = 2$.

$f'(x) = x^{\frac{1}{5}} - \dfrac{1}{x}$

$f(x) = \dfrac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1} - \ln|x| + C$

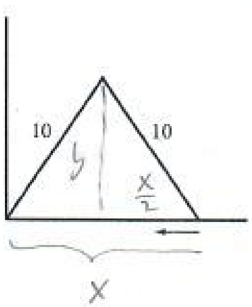
$= \dfrac{5}{6} x^{\frac{6}{5}} + \ln|x| + C$

$2 = f(1) = \dfrac{5}{6} \cdot 1^{\frac{6}{5}} + \underbrace{\ln|1|}_{=0} + C$

$2 = \dfrac{5}{6} + C, \quad C = 2 - \dfrac{5}{6} = \dfrac{7}{6}$

$f(x) = \dfrac{5}{6} x^{\frac{6}{5}} + \ln x + \dfrac{7}{6}$

13. (12pts) A folding ladder whose sides are 10ft long has one end against a wall. If the other end is pushed toward the wall at rate 1/2 foot per second, how fast is the top of the ladder rising when the pushed end is 12 feet away from the wall?



Know: $x' = -\frac{1}{2}$ ft/s, Need y' when $x = 12$

$\left(\dfrac{x}{2}\right)^2 + y^2 = 10^2$

$\dfrac{x^2}{4} + y^2 = 100 \quad \frac{d}{dx}$

$\dfrac{2xx'}{4} + 2yy' = 0$

$2yy' = -\dfrac{xx'}{2}$

$y' = -\dfrac{xx'}{4y}$

When $x = 12$

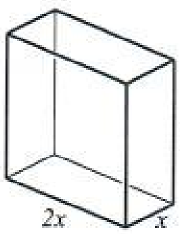
$\left(\dfrac{12}{2}\right)^2 + y^2 = 100$

$36 + y^2 = 100$

$y^2 = 64, \ y = \pm 8, \ y = 8$

$y' = -\dfrac{12 \cdot (-\frac{1}{2})}{4 \cdot 8} = \dfrac{6}{32} = \dfrac{3}{16}$ ft/s

14. (18pts) A rectangular box has a base whose length is twice the width. It uses 60 square inches of material (that is its surface area, including top and bottom). Find the dimensions x, y of the box that give the maximal possible volume of the box. (To get upper bound of the interval of consideration, notice that the sum of areas of the top and bottom sides cannot exceed 60.)



$S = 2\cdot 2x\cdot x + 2\cdot x\cdot y + 2\cdot 2x\cdot y = 4x^2 + 6xy = 60$

$6xy = 60 - 4x^2$

$y = \frac{60-4x^2}{6x} = \frac{30-2x^2}{3x}$

$V = 2x\cdot x\cdot y = 2x^2 y$

$= 2x^2\left(\frac{30-2x^2}{3x}\right)$

$= \frac{2}{3}(30x - 2x^3)$ ← Job: maximize $V(x)$ on $(0, \sqrt{15}]$

$2\cdot 2x\cdot x \le 60$

$4x^2 \le 60$

$x^2 \le 15$

$|x| \le \sqrt{15}$

So domain is

$(0, \sqrt{15}]$

Crit. pts: $V'(x) = 0$

$V'(x) = \frac{2}{3}(30 - 6x^2)$

$30 - 6x^2 = 0$

$6x^2 = 30$

$x^2 = 5$

$x = \pm\sqrt{5} = \sqrt{5}$

$V''(x) = \frac{2}{3}(-12x) = -8x$

$V''(\sqrt{5}) = -8\sqrt{5} < 0$

so V has a local max at $x = \sqrt{5}$

Since there is only one critical point

V has an absolute max at $x = \sqrt{5}$

$y = \frac{30 - 2\cdot\sqrt{5}^2}{3\sqrt{5}} = \frac{20}{3\sqrt{5}} = \frac{4\sqrt{5}}{3}$

Bonus. (10pts) Take the first five derivatives of $y = x\cos x$. Identify the pattern and use it to determine $y^{(59)}$.

$y = x\cos x$

$y' = 1\cdot\cos x + x(-\sin x) = \cos x - x\sin x$

$y'' = -\sin x - (1\cdot\sin x + x\cos x) = -2\sin x - x\cos x$

$y''' = -2\cos x - (1\cdot\cos x + x(-\sin x)) = -3\cos x + x\sin x$

$y^{(4)} = -3(-\sin x) + 1\cdot\sin x + x\cos x = 4\sin x + x\cos x$

$y^{(5)} = 4\cos x + 1\cdot\cos x + x(-\sin x) = 5\cos x - x\sin x$

Pattern is $f^{(k)}(x) = k\left(\pm \begin{smallmatrix}\sin x\\ \cos x\end{smallmatrix}\right) \pm x \begin{smallmatrix}\cos x\\ \sin x\end{smallmatrix}$

$59 = 14\cdot 4 + 3$ so signs will be like for 3rd derivative

$y^{(59)} = -59\cos x + x\sin x$