**Calculus 1 — Exam 4**
**MAT 250, Fall 2025 — D. Ivanšić**

Name: Saul Ocean

*Show all your work!*

**1.** (32pts) Let $f(x) = \dfrac{x^2}{e^x}$. The domain of this function is all real numbers (you do not have to verify this). Draw an accurate graph of $f$ by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
d) Use information from a)–c) to sketch the graph. If you run into some difficult $y$-values, you don't need to compute them, just mark a plausible point on the picture.

$$f'(x) = \frac{2xe^x - x^2e^x}{(e^x)^2} = \frac{e^x(2x-x^2)}{(e^x)^2} = \frac{2x-x^2}{e^x}$$

$$f''(x) = \frac{(2-2x)e^x - (2x-x^2)e^x}{(e^x)^2} = \frac{e^x(2-2x-2x+x^2)}{(e^x)^2} = \frac{x^2-4x+2}{e^x}$$

a) $f'(x)$ always defined

$f'(x)=0$: $2x-x^2=0$

$x(2-x)=0$

$x=0, 2$

$e^x>0$ so sign of $f'$ only depends on $2x-x^2$

$-$ 0 + 2 $-$

| | 0 | | 2 | |
|---|---|---|---|---|
| $f'$ | $-$ 0 | + | 0 $-$ | |
| $f$ | ↘ loc min | ↗ | loc min | ↘ |

b) $f''(x)$ always defined

$f''(x)=0$: $x^2-4x+2=0$

$$x = \frac{-(-4)\pm\sqrt{(-4)^2-4\cdot1\cdot2}}{2\cdot1} = \frac{4\pm\sqrt{16-8}}{2} = \frac{4\pm\sqrt{8}}{2} = \frac{4\pm2\sqrt{2}}{2}$$

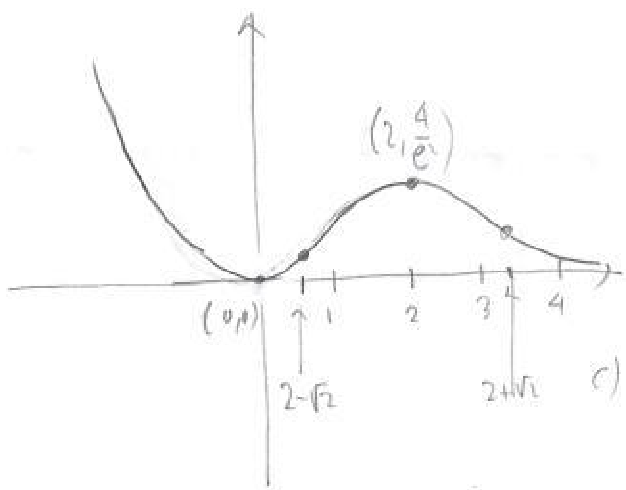
$= 2\pm\sqrt{2}$

sign of $f''$ depends only on $x^2-4x+2$

+ $2-\sqrt{2}$ $-$ $2+\sqrt{2}$ +

| | $2-\sqrt{2}$ | | $2+\sqrt{2}$ | |
|---|---|---|---|---|
| $f''$ | + 0 | $-$ | 0 + | |
| $f$ | CU IP | CD | IP CU | |

| $x$ | $\frac{x^2}{e^x}$ |
|---|---|
| 0 | 0 |
| 2 | $\frac{4}{e^2}$ |
| $2-\sqrt{2}$ | $\frac{(2-\sqrt{2})^2}{e^{2-\sqrt{2}}}$ |
| $2+\sqrt{2}$ | $\frac{(2+\sqrt{2})^2}{e^{2+\sqrt{2}}}$ |




c) $\lim\limits_{x\to\infty} \frac{x^2}{e^x} = \left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{2x}{e^x} = \left[\frac{\infty}{\infty}\right]$

$\overset{L'H}{=} \lim\limits_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$

$\lim\limits_{x\to-\infty} \frac{x^2}{e^x} = \frac{(-\infty)^2}{e^{-\infty}} = \frac{\infty}{0+} = \infty\cdot\infty = \infty$

**2.** (18pts) Let $f(x) = \frac{1}{4}\sin^2\theta + \frac{1}{3}\sin^3\theta$. Find the absolute minimum and maximum values of $f$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$f'(x) = \frac{1}{4}2\sin\theta\cos\theta + \frac{1}{3}\cdot 3\sin^2\theta\cos\theta$$

$$= \sin\theta\cos\theta\left(\frac{1}{2} + \sin\theta\right)$$

$f'(\theta)$ always defined

$f'(\theta) = 0$ when $\sin\theta = 0$ $\quad \cos\theta = 0$ $\quad \frac{1}{2} + \sin\theta = 0$

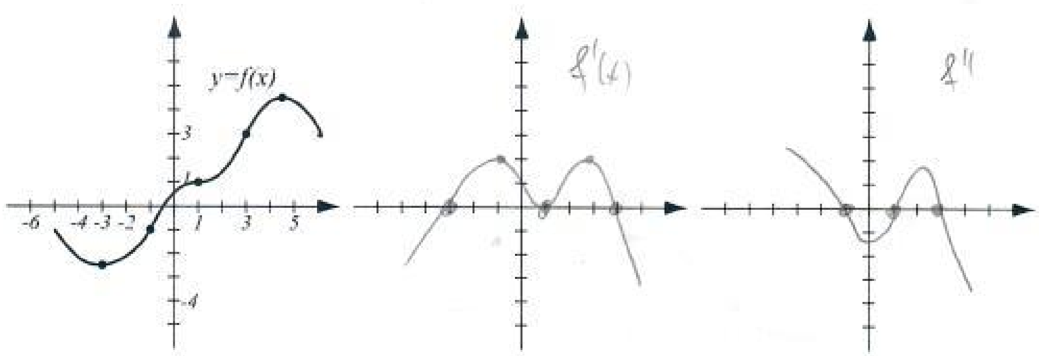
$\sin\theta = -\frac{1}{2}$

$\theta = 0, \pi$ $\quad \theta = -\frac{\pi}{2}, \frac{\pi}{2}$ $\quad \theta = -\frac{5\pi}{6}, -\frac{\pi}{6}$

not in interval

| $\theta$ | $\frac{1}{4}\sin^2\theta + \frac{1}{3}\sin^3\theta$ | |
|---|---|---|
| $-\frac{\pi}{2}$ | $\frac{1}{4}(-1)^2 + \frac{1}{3}(-1)^3 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$ | abs min |
| $\frac{\pi}{2}$ | $\frac{1}{4}\cdot 1^2 + \frac{1}{3}\cdot 1^3 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$ | abs max |
| $0$ | $0 + 0 = 0$ | |
| $-\frac{\pi}{6}$ | $\frac{1}{4}\left(-\frac{1}{2}\right)^2 + \frac{1}{3}\left(-\frac{1}{2}\right)^3 = \frac{1}{16} - \frac{1}{24} = \frac{1}{48}$ | |

**3.** (14pts) The graph of $f$ is given. Use it to draw the graphs of $f'$ and $f''$ in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of $f$. The relevant special points have been highlighted.



**4.** (14pts) Consider $f(x) = \sqrt{x}$ on the interval $[1, 9]$.
a) Verify that the function satisfies the assumptions of the Mean Value Theorem.
b) Find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.

a) $f$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$
so cont. on $[1,9]$ and diff. on $(1,9)$

b) $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ for some $c$ in $(a,b)$

$$\frac{f(9)-f(1)}{9-1} = \frac{\sqrt{9}-\sqrt{1}}{9-1} = \frac{3-1}{8} = \frac{1}{4}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

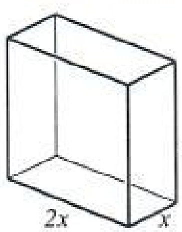
$$\frac{1}{2\sqrt{x}} = \frac{1}{4}$$

$2\sqrt{x} = 4$

$\sqrt{x} = 2$

$x = 4$, which is in $(1,9)$

**5.** (22pts) A rectangular box has a base whose length is twice the width. It uses 60 square inches of material (that is its surface area, including top and bottom). Find the dimensions $x$, $y$ of the box that give the maximal possible volume of the box. (To get upper bound of the interval of consideration, notice that the sum of areas of the top and bottom sides cannot exceed 60.)




Surface area is $2\cdot 2x\cdot x + 2\cdot 2x\cdot y + 2\cdot x\cdot y = 4x^2+6xy$

$4x^2+6xy=60$

$6xy = 60-4x^2$

$$y = \frac{60-4x^2}{6x} = \frac{30-2x^2}{3x}$$

$$V = 2x\cdot x\cdot y = 2x^{\cancel{2}}\cdot\frac{30-2x^2}{3\cancel{x}}$$

$$V(x) = \frac{2}{3}(30x-2x^3)$$

Job: maximize $V(x)$ on $(0, \sqrt{15}]$

Must have
$2\cdot 2x\cdot x \le 60$
$x^2 \le 15$
$x \le \sqrt{15}$
$x \geq 0$

$V'(x) = \frac{2}{3}(30-6x^2)$ always defined

$V'(x)=0$: $30-6x^2=0$
$6x^2=30$
$x^2=5$
$x=\pm\sqrt{5}$, $x=\sqrt{5}$

$V''(x) = \frac{2}{3}(-12x) = -8x$

$V''(\sqrt{5}) = -8\sqrt{5} < 0$

There is a local max at $x=\sqrt{5}$, since there is only one critical point, it is the absolute max.

Dimensions: $2\sqrt{5}$ by $\sqrt{5}$ by $\frac{20}{3\sqrt{5}} \leftarrow \frac{30-2\sqrt{5}^2}{3\sqrt{5}} = \frac{20}{3\sqrt{5}}$

**Bonus.** (10pts) Function $f(x)$ is such that $f(1) = -3$ and $f'(x) = 3\cos^2\dfrac{1}{x} + 1$. Use the Mean Value Theorem to show that $1 \le f(5) \le 13$.

$$-1 \le \cos\frac{1}{x} \le 1$$

$$0 \le \cos^2\frac{1}{x} \le 1$$

$$0 \le 3\cos^2\frac{1}{x} \le 3$$

$$1 \le 3\cos^2\frac{1}{x} + 1 \le 4$$

Thus, $1 \le f'(x) \le 4$ for all $x$

By MVT, there is a $c$ in $(1,5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5-1}$$

Since $1 \le f'(c) \le 4$, we get

$$1 \le \frac{f(5) - f(1)}{5-1} \le 4$$

$$1 \le \frac{f(5) - (-3)}{4} \le 4$$

$$4 \le f(5) + 3 \le 16$$

$$1 \le f(5) \le 13$$