Calculus 1 — Exam 3
MAT 250, Fall 2025 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (5pts) $\frac{d}{dx}(x^2-3x+2)e^x = (2x-3)e^x+(x^2-3x+2)e^x$

$= (x^2-x-1)e^x$

2. (5pts) $\frac{d}{dz}6^{\arctan z} = \ln 6 \cdot 6^{\arctan z} \cdot \frac{1}{1+z^2} = \frac{\ln 6\, 6^{\arctan z}}{1+z^2}$

3. (7pts) $\frac{d}{du}\frac{u\ln u}{\ln u+1} = \frac{(1\cdot \ln u + u\cdot\frac{1}{u})(\ln u+1) - u\ln u\cdot\frac{1}{u}}{(\ln u+1)^2} = \frac{(\ln u+1)(\ln u+1)-\ln u}{(\ln u+1)^2}$

$= \frac{(\ln u)^2+2\ln u+1-\ln u}{(\ln u+1)^2} = \frac{(\ln u)^2+\ln u+1}{(\ln u+1)^2}$

4. (7pts) $\frac{d}{dx}\ln\frac{1+\tan x}{1-\tan x} = \frac{d}{dx}\left(\ln(1+\tan x)-\ln(1-\tan x)\right) = \frac{1}{1+\tan x}\cdot\sec^2 x - \frac{1}{1-\tan x}\sec^2 x$

$= \sec^2 x\left(\frac{1}{1+\tan x}+\frac{1}{1-\tan x}\right) = \sec^2 x\,\frac{1-\tan x+1+\tan x}{(1+\tan x)(1-\tan x)} = \frac{2\sec^2 x}{1-\tan^2 x}$

5. (7pts) $\frac{d}{dx}x\arcsin\sqrt{x} = 1\cdot\arcsin\sqrt{x} + x\cdot\frac{1}{\sqrt{1-\sqrt{x}^2}}\cdot\frac{1}{2\sqrt{x}} = \arcsin x + \frac{x}{\sqrt{1-x}\,2\sqrt{x}}$

$= \arcsin x + \frac{\sqrt{x}}{2\sqrt{1-x}}$

6. (9pts) Use logarithmic differentiation to find the derivative of $y=(\sin x)^{x^2+2x}$.

$y=(\sin x)^{x^2+2x}$

$\frac{d}{dx}$ $\quad \ln y = (x^2+2x)\ln\sin x$

$\frac{1}{y}\cdot y' = (2x+2)\ln\sin x + (x^2+2x)\frac{1}{\sin x}\cos x \quad |\cdot y$

$y' = \left((2x+2)\ln\sin x + (x^2+2x)\cot x\right)(\sin x)^{x^2+2x}$

Find the limits algebraically. Graphs of basic functions will help, as will L'Hospital's rule, where appropriate.

7. (2pts) $\lim_{x\to\infty} \arctan(7x) =$ $\arctan(7\cdot\infty) = \arctan\infty = \frac{\pi}{2}$

8. (6pts) $\lim_{x\to\infty} e^{-\frac{x^2+1}{x+3}} =$ $e^{\lim_{x\to\infty} -\frac{x^2+1}{x+3}} = e^{\lim_{x\to\infty} -\frac{x^2(1+\frac{1}{x^2})}{x(1+\frac{3}{x})}} = e^{\lim_{x\to\infty} -x\cdot\left(\frac{1+0}{1+0}\right)}$

$= e^{-\infty\cdot 1} = e^{-\infty} = 0$

9. (6pts) $\lim_{x\to\infty} \frac{\ln x}{\sqrt[10]{x}} =$ $\left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{10}x^{-\frac{9}{10}}} = \lim_{x\to\infty} 10x^{-1-(-\frac{9}{10})} = \lim_{x\to\infty} 10x^{-\frac{1}{10}}$

$x^{1/10}$

$= \lim_{x\to\infty} \frac{10}{\sqrt[10]{x}} = \frac{10}{\infty} = 0$

10. (9pts) $\lim_{x\to 0} \frac{5x - \sin(5x)}{x^3} =$ $\left[\frac{5\cdot 0 - \sin(5\cdot 0)}{0^3} = \frac{0}{0}\right] \overset{L'H}{=} \lim_{x\to 0} \frac{5 - \cos(5x)\cdot 5}{3x^2} = \left[\frac{5-1\cdot 5}{3\cdot 0} = \frac{0}{0}\right]$

$\overset{L'H}{=} \lim_{x\to 0} \frac{25\sin(5x)}{6x} = \left[\frac{0}{0}\right] \overset{L'H}{=} \lim_{x\to 0} \frac{125\cos(5x)}{6} = \frac{125}{6}\cdot\underbrace{\cos 0}_{=1} = \frac{125}{6}$

11. (8pts) $\lim_{x\to 0} \underbrace{(1-4x)^{\frac{2}{x}}}_{y} =$ $\lim_{x\to\infty} e^{\ln y} = e^{-8}$

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \ln(1-4x)^{\frac{2}{x}} = \lim_{x\to 0} \frac{2\ln(1-4x)}{x} = \left[\frac{2\ln 1}{0} = 0\right] \overset{L'H}{=} \lim_{x\to 0} \frac{2\frac{1}{1-4x}(-4)}{1}$

$= \lim_{x\to 0} \frac{-8}{1-4x} = \frac{-8}{1-0} = -8$

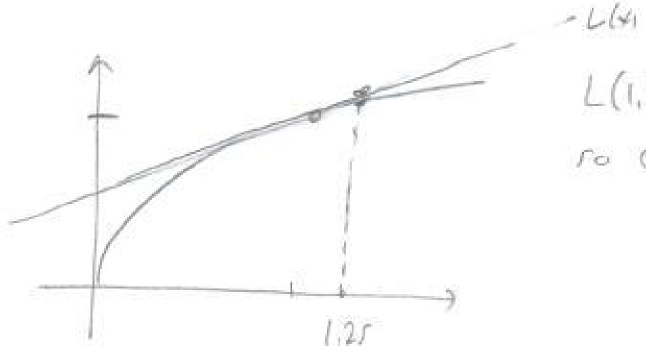
12. (11pts) Let $f(x) = \sqrt{x}$.
a) Write the linearization of $f(x)$ at $a = 1$.
b) Use the linearization to estimate $\sqrt{1.25}$.
c) In the same coordinate system, draw graphs of the function and the linearization and determine if the estimate in b) is an overestimate or underestimate of $\sqrt{1.25}$.

a) $L(x) = f(1) + f'(1)(x-1)$ $\qquad f'(x) = \frac{1}{2\sqrt{x}}$

$\sqrt{1} = 1 \qquad \frac{1}{2\sqrt{1}} = \frac{1}{2}$

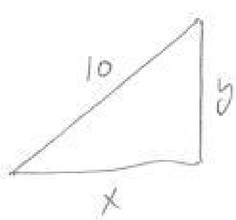
$L(x) = 1 + \frac{1}{2}(x-1) = \frac{1}{2}x + \frac{1}{2}$

b) $\sqrt{1.25} \approx L(1.25) = L(\frac{5}{4}) = \frac{1}{2}\cdot\frac{5}{4} + \frac{1}{2} = \frac{5}{8} + \frac{1}{2} = \frac{9}{8} = 1.125$

c) 

$L(1.25) > \sqrt{1.25}$
so estimate is an overestimate

13. (10pts) In a right triangle, the hypothenuse is known to be 10 meters long. One of the sides is measured to be 6 meters, with maximum error 2cm. Use differentials to estimate the maximum possible error when computing the length of the other side of the triangle.



$x^2 + y^2 = 10^2$

$y^2 = 10^2 - x^2$

$y = \pm\sqrt{100 - x^2}$

$y = \sqrt{100 - x^2}$ since $y > 0$

$\Delta y \approx dy = f'(x)\,dx = \frac{1}{2\sqrt{100-x^2}}\cdot(-2x)\,dx = -\frac{x}{\sqrt{100-x^2}}\,dx$

when $x = 6$ m $\quad -\frac{6}{\sqrt{100-6^2}}\cdot 0.02 = -\frac{6}{\sqrt{64}}\cdot 0.02 = -\frac{\cancel{6}^{3}}{\cancel{8}_{2\cdot 4}}\cdot 0.0\cancel{2}^{1}$

$dx = 2\text{cm} = 0.02$ m

$= -\frac{0.03}{2} = -0.015$ m $\quad$ (1.5 cm)

14. (8pts) Let $f(x) = x^5$. Use the theorem on derivatives of inverses to establish the formula for the derivative of $\sqrt[5]{x}$ and see that you get the same expression as using usual rules.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt[5]{x})} = \frac{1}{5(\sqrt[5]{x})^4} = \frac{1}{5x^{\frac{4}{5}}} = \frac{1}{5}x^{-\frac{4}{5}} = \frac{d}{dx}x^{\frac{1}{5}}$$

$f(x) = x^5$

$f'(x) = 5x^4$

$f^{-1}(x) = \sqrt[5]{x}$

which is the usual rule

Bonus. (10pts) Find the derivative and simplify until the bitter end. You will get the derivative of a simpler function. Which one?

$$\frac{d}{dx}\arcsin\sqrt{\frac{1-x}{2}} = \frac{1}{\sqrt{1-\sqrt{\frac{1-x}{2}}^2}} \cdot \frac{1}{2\sqrt{\frac{1-x}{2}}} \cdot \left(-\frac{1}{2}\right) = -\frac{1}{\sqrt{1-\frac{1-x}{2}}} \cdot \frac{1}{4\sqrt{\frac{1-x}{2}}}$$

$$= -\frac{1}{\sqrt{\frac{2-(1-x)}{2}}} \cdot \frac{1}{4\sqrt{\frac{1-x}{2}}} = -\frac{1}{4\sqrt{\frac{1+x}{2}}\sqrt{\frac{1-x}{2}}} = -\frac{1}{4\sqrt{\frac{1-x^2}{4}}}$$

$$= -\frac{1}{\frac{4\sqrt{1-x^2}}{2}} = -\frac{1}{2\sqrt{1-x^2}} = \frac{d}{dx}\left(\frac{1}{2}\arccos x\right)$$