Calculus 1 — Lecture notes
MAT 250, Fall 2025 — D. Ivanšić

2.1 Derivatives and Rates of Change

Example. Suppose you were in Murray at 1PM and Paducah at 2PM, having driven a distance of 50 miles. How fast were you going, on average?

We say the *average velocity* was 50 mi/hr.

Definition. Let $s = f(t)$ be the position of an object at time t.

$$\begin{matrix}\textit{average velocity} \\ \textit{over interval } [a, a+h]\end{matrix} = \frac{\text{change in position}}{\text{change in time}} = \frac{f(a+h) - f(a)}{h}$$

From daily experience we know the idea of *instantaneous velocity* makes sense, and that it depends on the moment of observation. For example:

— The speedometer on the car varies, purportedly showing the *current* speed.

— On our way from Murray to Paducah a police officer will not stop us based on how long it would take to travel to Paducah, but based on *how fast* we were going *when* they observed us.

— How much it hurts when a baseball hits us does not depend from how far away it came or how long it took to reach us, but on *how fast* it was flying *when* it hit us.

Try to define what instantaneous velocity is.

You probably struggled to define it without using the same terms. The proper question is: how do you compute instantaneous velocity?

Instantaneous velocity at time $t = a$ ought to be close to average velocity over the time interval $[a, a+h]$ for a short time interval, that is, a small h. The shorter the time interval, the closer the average velocity of over that time interval ought to be to instantaneous velocity. Therefore, it makes sense to define:

Definition. Let $s = f(t)$ be the position of an object at time t. We define

$$v(a) = \begin{matrix}\textit{instataneous velocity} \\ \textit{at time } t = a\end{matrix} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \begin{matrix}\text{limit of average velocities} \\ \text{over } [a, a+h]\text{, as } h \to 0\end{matrix}$$

Note. When $h < 0$, the interval is actually $[a+h, a]$ but the limit stays the same.

Example. It was established experimentally that, measured with respect to point of release, the position of a stone dropped from a building is $f(t) = 5t^2$ meters, where t is in seconds. Find $v(3)$.

Example. It is intuitively clear that a smooth curve has a *tangent line*, a line that fits the curve snugly. For example, if a curve is represented by a wire, there is really only one position in which a ruler touches a curve at a point.

Example. Find the equation of the tangent line to the curve $y = x^2$ at the point $P = (2, 4)$.

Idea: take a point Q on the curve, $Q \neq P$, and consider the slope of the line PQ, called a *secant line*. As Q slides toward P, the slope of the secant line ought to get closer and closer to the slope of the tangent line.

Definition. The *tangent line* to graph of $f(x)$ at point $(a, f(a))$ is the line through point $(a, f(a))$ whose slope is:

$$\begin{array}{c}\textit{slope of tangent line}\\ \textit{at point } (a, f(a))\end{array} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \begin{bmatrix}\text{set}\\ x = a+h\end{bmatrix} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

The ideas above, where the expression $\frac{f(x)-f(a)}{x-a}$ comes up repeatedly, can be used to define a general concept for any function $f(x)$:

Definition.

$$\begin{array}{c}\textit{average rate of change}\\ \textit{of } f \textit{ from } a \textit{ to } x\end{array} = \frac{f(x)-f(a)}{x-a} = \frac{\Delta f}{\Delta x} = \frac{\Delta y}{\Delta x} = \frac{\text{change in } f}{\text{change in } x}$$

$$\begin{array}{c}\textit{instataneous rate of change}\\ \textit{of } f \textit{ at } a\end{array} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \begin{array}{l}\text{limit of average rates of change}\\ \text{over } [a, x]\text{, as } x \to a\end{array}$$

When some expression repeatedly shows up in applications, it gets abstracted to a mathematical notion.

Definition. The *derivative* of f at a is

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h},$$

if this limit exists.

Interpretation. The derivative $f'(a)$ represents the

— instantaneous rate of change of a function f with respect to x at point a
— slope of tangent line to graph of f at point $(a, f(a))$
— instantaneous velocity at time a of an object whose position is described by function f

Example. As found in example above, if $f(t) = 5t^2$, $f'(3) = 30$.
At time $t = 3$, instantaneous velocity of the falling stone is 30 m/s.

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2.2 The Derivative as a Function

The derivative has so far been defined at a point a: the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For different a's we'll get different values, which gives rise to a function f', for which we can say:

$$f'(x) = \text{ slope of tangent line to graph of } f \text{ at } (x, f(x)).$$

Example. Let $f(x) = x^3$.
Roughly draw the graph of $f'(x)$.

Now calculate $f'(a)$ for a general a.

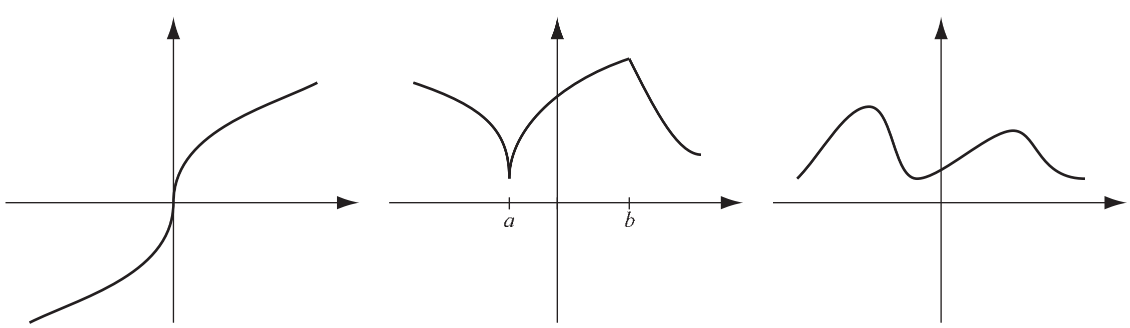
Example. Let $f(x) = \sqrt{x}$.
Roughly draw the graph of $f'(x)$.

Now calculate $f'(a)$ for a general a.

Example. Let $f(x) = |x|$.
Draw the graph of $f'(x)$ and write its formula.

Definition. A function f is *differentiable at a* if $f'(a)$ exists. A function f is *differentiable on an open interval* if $f'(a)$ exists at every point a of the interval.

Example. At which points are the functions below not differentiable?



Note. A function f fails to be differentiable at points where

— the graph of f has a "sharp point"

— the tangent line to graph of f is vertical

— where the function is not continuous

Theorem. If a function f is differentiable at a, then it is continuous at a.

Proof.

Notation for a derivative:

$$f'(x) = y'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)$$

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \begin{bmatrix} \text{stands for} \\ f'(x) \text{ evaluated at } a \end{bmatrix} = f'(a)$$

Note. Even though the notation looks like a fraction, the derivative is not a fraction. The notation evokes the definition, which uses a fraction.

The $\frac{d}{dx}$ notation is especially useful in situations where more than one letter is in the function. It identifies the letter that we are treating as the variable in the function, and we treat the other letters as constants.

Example. $\frac{d}{dx}(zx^2 + z^2x + z^3)$

Higher derivatives. If f is differentiable, we have the function f'. If now f' has a derivative, it is called the second derivative of f; we write $(f')' = f''$. Similarly, the derivative of f'' is $(f'')' = f'''$, called the third derivative of f. For higher derivatives than the fourth, we usually write $f^{(n)}$ for the n-th derivative of f.

Example. Let $s(t)$ be the position function of an object. Then

— $v(t) = s'(t)$, velocity is rate of change of position

— $a(t) = v'(t) = s''(t)$, acceleration is rate of change of velocity

2.3 Basic Differentiation Formulas

We start with derivatives of the most basic functions:

$$\frac{d}{dx}\,c = 0 \qquad\qquad \frac{d}{dx}\,x = 1$$

Example. Find $f'(x)$, if $f(x) = x^4$.

The Power Rule. For an integer $n, n > 0$, we have:

$$\frac{d}{dx}\,x^n = nx^{n-1}$$

This is true for a general exponent (negative or fractional). For example,

$$\text{for } f(x) = \sqrt{x}, \text{ we have found } f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$$

Theorem. If f and g are differentiable functions and c is a constant, then

$$(cf(x))' = cf'(x) \qquad\qquad (f(x) \pm g(x))' = f'(x) \pm g'(x)$$

Example. $\dfrac{d}{dx}\left(x^3 - 2x^2 + 7x - 1\right) =$

Example. $\frac{d}{dx}\left(\frac{1}{x^3} - 7\sqrt[4]{x^5} + x\sqrt{x}\right) =$

Example. $\frac{d}{dx}\frac{x^3 - 4x + 1}{\sqrt{x}} =$

Derivatives of $\sin x$, $\cos x$

$$\frac{d}{dx}\sin x = \cos x \qquad\qquad \frac{d}{dx}\cos x = -\sin x$$

Proof for $\sin x$.

Example. Find $(\sin x)^{(45)}$.

Applications of rates of change

Example. Suppose that the cost of producing x jars of homemade jam is given by $C(x) = 2200 + 0.16x - 0.006x^2 + 0.00015x^3$.
a) Find the marginal cost function.
b) Using $C'(100)$, estimate the cost of producing the 101st jar.
c) Find the actual cost of producing the 101st jar.

Example. The general formula for the position of an object released from height s_0 at velocity v_0 is $s(t) = -\frac{g}{2}t^2 + v_0 t + s_0$, where g is the gravitational constant (9.8≈10m/s^2 in the metric system, 32ft/s^2 in the English system). Suppose an arrow is shot up with initial velocity 20 meters per second.
a) Find the formula for the velocity of the arrow.
b) When does the arrow hit the ground? With what velocity?
c) When does it reach the highest altitude and what is this altitude?
d) When does it reach height 10m?

Example. Suppose the position of an object is given by $s(t) = t^3 - 6t^2 + 9t$.
a) Find $v(t)$, $a(t)$ and graph these functions.
b) When is the object moving forward? Backward? At rest?
c) Illustrate motion of the particle with a diagram.
d) Find total distance traveled from $t = 0$ to $t = 5$.
e) When is the object speeding up? Slowing down?

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2.4 The Product and Quotient Rules

Use examples to illustrate the following:

$$(f(x)\cdot g(x))' \neq f'(x)\cdot g'(x) \qquad \left(\frac{f(x)}{g(x)}\right)' \neq \frac{f'(x)}{g'(x)}$$

The Product and Quotient Rules

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \qquad \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Example. $\dfrac{d}{dx}\left((x^3 - 2x^2 + 1)(3x^2 + 2)\right) =$

Example. $(\sqrt{t}\sin t)' =$

Example. $\dfrac{d}{dx}\dfrac{x^2+1}{x^3-x} =$

Example. $\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right)' =$

Example. $\dfrac{d}{dx}\tan x =$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

Note. See the similarities between derivatives of sec, csc, tan and cot, and note that the derivative of every function whose name starts with "co" has a minus.

A justification of the product rule.

2.5 The Chain Rule

Example. Find the derivative of the function $\sqrt{x^3+5x^2+4}$. This cannot be done using any of the rules we had so far, as this function is a composite $f(g(x))$ of functions

$$g(x) = x^3 + 5x^2 + 4 \text{ and } f(x) = \sqrt{x}$$

We need a rule for differentiating composites of functions.

The Chain Rule. If f and g are differentiable, then $f \circ g$ is differentiable and

$$(f \circ g)'(x) = (f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Example. $\dfrac{d}{dx}\sqrt{x^3+5x^2+4} =$

Example. $\dfrac{d}{dx}(3x+1)^{10} =$

$\dfrac{d}{dx}\sin\left(2x+\dfrac{\pi}{4}\right) =$

Example. $\dfrac{d}{dx}\sin^3 x =$

$\dfrac{d}{dx}\sin(x^3) =$

Example. $\dfrac{d}{ds}(s^2-7s+3)^{20} =$

Example. $\dfrac{d}{dx}\left((3x-5)^2\cos x\right) =$

Example. $\dfrac{d}{dt}\sqrt[3]{\dfrac{t}{t^2+1}} =$

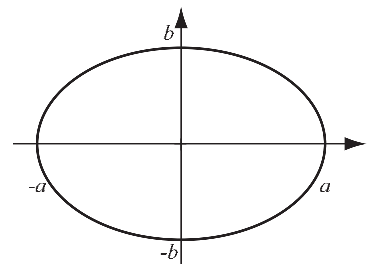
Example. $\dfrac{d}{du}\tan\sqrt{u^4-5u^2+1} =$

Another way to write the chain rule. If $y = f(u)$ and $u = g(x)$, then y is a function of x via u, and we can write

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Example. (Why the chain rule works the way it does.) A diver descends in the ocean at rate $\frac{1}{2}$ ft/s, while pressure increases 0.434 lb/in^2 for every foot of depth. How fast is pressure changing for the diver?

2.6 Implicit Differentiation



Example. An ellipse with semiaxes a and b is the set of all points (x, y) in the plane that satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Note that for $a = b = r$ we get the equation of a circle.

Example. Find the tangent line to the ellipse $\frac{x^2}{4} + \frac{y^2}{12} = 1$ at point $(1, -3)$.

First solution: solve for y.

Second solution: use the method of *implicit differentiation*.
Considering only the part of the ellipse near the point $(1, -3)$, it is the graph of some function $y(x)$ which satisfies the equation of the ellipse, where we write $y(x)$ instead of y. Differentiate both sides of the equation by x.

Example. The folium of Descartes is the curve whose equation is $x^3 + y^3 = 6xy$.
a) Find the slope of the tangent line at $(3, 3)$.
b) Where is the tangent line horizontal?

Example. Find y' if the function $y(x)$ satisfies the equation.

$$x \sin y + \cos y^2 = \frac{x}{y}$$

2.7 Related Rates

Example. At noon, cars A and B are at a crossroads. Car A heads out east at 50mph right away, while car B waits until 1PM, then heads out north at 75mph. At what rate is the distance between the two cars increasing at 2PM?

Example. A light house is on a small island 3km away from the nearest point P on a straight shoreline and its light makes 4 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1km from P?

Example. A trough is 10ft long and its ends are the shape of an isosceles triangle that is 3ft across at the top and 1ft high. If the trough is being filled at the rate of 12ft^3/min, how fast is the water level rising when depth is 6 inches?

Methods of attack of these problems:

1) Read the problem carefully.
2) Draw the @!♣#◇$ picture! Large, clear with indicated quantities.
3) Understand which quantities are changing and assign them symbols. Quantities that are not changing can be written as their values.
4) Write, usually in terms of derivatives, what you need and what you know.
5) Get an equation from the picture. The equation typically uses basic geometry.
6) Differentiate the equation. The chain rule is frequently needed.
7) Express what you need in terms of what you know.
8) Only now plug in the numbers given for the quantities.

2.8 Linear Approximation and Differentials

The *linear approximation (linearization)* of a function f at a point a is simply the linear function $L(x)$ representing the tangent line throught the point $(a, f(a))$:

$$L(x) = f(a) + f'(a)(x - a)$$

For numbers close close to a, $L(x)$ is a pretty good approximation of $f(x)$.

Note. $L(a) = f(a)$ and $L'(a) = f'(a)$; functions L and f have the same value and first derivative at a, therefore, they should be close near a.

Example. Use linearization to approximate $\sqrt[3]{8.2}$, $\sqrt[3]{7.9}$.

Now use a graphing calculator to determine for which interval the linear approximation of $\sqrt[3]{x}$ at 8 has accuracy 0.01, respectively 0.001.

Linearization can be used to approximate the change in the function. For this purpose, the language of *differentials* dy and dx is used.

$dx = \Delta x =$ change in x
$dy =$ change in y of linearization
$\Delta y =$ change in y of function

For a small Δx, $\Delta y \approx dy$, and to find dy, we use the equation

$$dy = f'(x)dx$$

Example. The radius of a sphere is measured as 15 cm with error in measurement 0.3 cm. What is the estimated maximum error (absolute and relative) of calculated
a) surface area of sphere
b) volume of sphere?