Trigonometry — Lecture notes
MAT 145, Fall 2025 — D. Ivanšić

8.1 Law of Sines

At the beginning of the course, we *solved* right triangles. This meant, given two independent pieces of information about a right triangle (a side and an angle, or two sides) we found all the sides and all the angles. (Note that the two acute angles in a right triangle are not independent due to $\beta = 90° - \alpha$.)

Now we wish to solve *oblique* triangles, the triangles that are not right.

all angles acute   one angle obtuse ($> 90°$)

Two shapes of oblique triangles are possible:

Note a triangle cannot have more than one obtuse angle, since otherwise the sum of all angles exceeds 180°.

We adopt the usual labeling standard, where angles A, B, C are opposite sides a, b, c, respectively.

For an oblique triangle, three independent pieces of information about the triangle are needed. (For a right triangle, it is really the same, it's just that the third piece of information is that one angle is 90°.) The following three pieces of information determine a triangle.

ASA
side and two angles adjacent to it

AAS
two angles and side opposite one of them

SSA
two sides and angle opposite one side

SAS
two sides and angle between them

SSS
three sides

Our first tool in solving triangles is:

Law of sines. In a triangle with angles A, B, C opposite sides a, b, c we have:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof.

Example. Solve the ASA triangle where $c = 5$, $A = 35°$, $B = 70°$. First explain why there is exactly one solution.

Example. Solve the AAS triangle where $a = 7$, $A = 45°$, $B = 40°$. First explain why there is exactly one solution.

Example. Solving an SSA triangle involves several possibilities:

1) Side opposite the angle is too short: no solution.

2) Side opposite the angle is just long enough to reach the base: one solution or none

3) Side opposite the angle is shorter than the adjacent side and reaches base in two places: two solutions or none.

4) Side opposite the angle is longer than the adjacent side: one solution.

Example. Solve the SSA triangle where $A = 40°$, $a = 2$, $b = 4$.

Example. Solve the SSA triangle where $C = 50°$, $a = 5$, $c = 4$.

Example. Solve the SSA triangle where $B = 60°$, $b = 7$, $c = 5$.

Area of a triangle. The area of a triangle equals half of product of lengths of two sides and the sine of the angle between them. (Note that all the variables are different letters.)

$$\text{Area} = \frac{1}{2}ab\sin C$$

Proof.

Example. Find the area of the triangle where $c = 8\,\text{in}$, $B = 38°$, $C = 123°$.

8.2 Law of Cosines

Note that the Law of Sines does not help with SAS or SSS triangles, buecause we get equations with more than one unknown.

Law of Cosines. In a triangle with angles A, B, C opposite sides a, b, c we have:

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad b^2 = a^2 + c^2 - 2ac\cos B \qquad a^2 = b^2 + c^2 - 2bc\cos A$$

Proof.

Example. Solve the SAS triangle where $a = 3$, $b = 4$, $C = 55°$. First explain why there is exactly one solution.

Example. Solve the SSS triangle where $a = 3$, $b = 7$, $c = 5$. First explain why there is exactly one solution, as long as the sum of lengths of every pair of sides is bigger than the third one.

Trigonometry — Lecture notes
MAT 145, Fall 2025 — D. Ivanšić

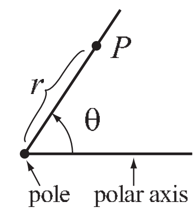
8.4 Polar Coordinates

To specify the position of a point in the plane, we have so far used rectangular (Cartesian) coordinates.

Fix a ray, called the *polar axis*, along which we will place the initial side of an angle, usually the positive x-axis. We may also specify the position of a point by giving

1) the angle θ on whose terminal side point is located.
2) the distance r from the origin.

We say (r, θ) are *polar coordinates* of the point P.



Example. Sketch the points with polar coordinates:

$\left(5, \frac{\pi}{2}\right)$ $\qquad$ $\left(3, -\frac{5\pi}{6}\right)$ $\qquad$ $(4, 180°)$

In polar coordinates, we allow r to be negative (so, r is "directed distance"). If $r < 0$ we go distance $|r|$ from the origin along the ray that is opposite to terminal side of angle θ.

Example. Sketch the points with polar coordinates:

$\left(-2, \frac{\pi}{2}\right)$ $\qquad$ $(-3, \pi)$ $\qquad$ $(-4, 30°)$

Example. In contrast to rectangular coordinates, every point has many polar coordinates. List all the possible coordinates of the point with polar coordinates $\left(4, \frac{\pi}{3}\right)$

Converting coordinates. Using formulas $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$ we get the conversion formulas at right.

polar $\to$ rectangular	rectangular $\to$ polar
$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$

Example. Convert to rectangular coordinates.

$\left(5, \dfrac{5\pi}{6}\right)$ $\qquad$ $(-3, 60°)$

Example. Convert to polar coordinates.

$(2,2)$ $\qquad$ $(-3, \sqrt{3})$ $\qquad$ $(-3,-7)$

Convert the following equations to polar coordinates.

Example. $x^2 + y^2 = 9$

Example. $y = 3x + 5$

Convert the following equations to rectangular coordinates.

Example. $r = \sin\theta$

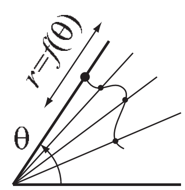
Example. $r = \sin(2\theta)$

Curves with basic equations in r, θ can be drawn fairly easily.

Example. Sketch the curves given by the following equations.

$r = 2$ $\qquad\qquad\qquad\qquad$ $\theta = \dfrac{\pi}{3}$

A polar equation typically has the form $r = f(\theta)$, so r varies as a function of θ. We can imagine the curve as the path of a bead moving back an forth along the terminal side of the angle as it circles around the origin.



Graph the following polar equations.

Example. $r = 1 + \sin\theta$, θ in $[0, 2\pi]$

Example. $r = \cos(2\theta)$, θ in $[0, 2\pi]$

Example. $r = 4\sin(3\theta)$, θ in $[0, 2\pi]$

$$\left.\begin{array}{l} r = a\sin(n\theta) \\ r = a\cos(n\theta) \end{array}\right\} \text{ is a rose with } \left\{\begin{array}{l} n \text{ petals, if } n \text{ is odd (traversed once for } 0 \le \theta \le \pi) \\ 2n \text{ petals, if } n \text{ is even (traversed once for } 0 \le \theta \le 2\pi) \end{array}\right.$$

Example. $r = \theta$, θ in $[0, \infty]$

Example. $r = f(\theta)$, where f is given by the graph below.