**College Algebra — Lecture notes**
**MAT 140, Fall 2025 — D. Ivanšić**

## 5.1 Inverse Functions

Consider these functions:

$f(x) = 2x$ If $a \neq b$, then $2a \neq 2b$

$f(x) = x^3$ If $a \neq b$, then $a^3 \neq b^3$

$f(x) = x^2$ $-1 \neq 1$, but $(-1)^2 = 1^2$

$f(x) = |x|$ $-2 \neq 2$, but $|-2| = |2|$

**Definition.** A function is *one-to-one*, if different $x$'s are sent to different $y$'s, that is

$$\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2)$$

**Example.** Draw graphs of $f(x) = x^3$ and $g(x) = |x|$. Can you tell from the graph whether they are one-to-one?

The pictures suggest the following test on graphs for one-to-one functions.

**Horizontal line test.** If every horizontal line intersects the graph of a function in at most one point, then the function is one-to-one.

**Example.** Which of the following graphs is one-to-one?

**Example.** The function $f(x) = 2x + 3$ is one-to-one. The reverse rule of $f$ is the one that sends every element of the range to the element sent to it via $f$. This function is called the *inverse of* $f$, and written as $f^{-1}$.

$$-2 \overset{f}{\longmapsto} \qquad\qquad \overset{f^{-1}}{\longmapsto} -2$$

$$0 \overset{f}{\longmapsto} \qquad\qquad \overset{f^{-1}}{\longmapsto} 0$$

$$2 \overset{f}{\longmapsto} \qquad\qquad \overset{f^{-1}}{\longmapsto} 2$$

$$4 \overset{f}{\longmapsto} \qquad\qquad \overset{f^{-1}}{\longmapsto} 4$$

**Example.** To find the formula for the inverse rule, write $y = 2x + 3$ and solve for $x$.

**Example.** Find the inverse formula for the function $f(x) = 5(x - 1)^3$.

**Note.**

| $f(x) =$ | $f$ does | undo this | $f^{-1}(y) =$ |
|---|---|---|---|
| 2x+3 | | | |
| $5(x-1)^3$ | | | |

Another way to view $f^{-1}$ as undoing what $f$ does is that their composite is the identity function.

$x \xmapsto{f} 2x + 3 \xmapsto{f^{-1}}$ Therefore, $f^{-1}(f(x)) = x$.

$x \xmapsto{f} 5(x-1)^3 \xmapsto{f^{-1}}$ Similarly, $f(f^{-1}(x)) = x$.

**Example.** Let $f(x) = x^2$. What is $f^{-1}(4)$?

$-2 \xmapsto{f}$

$-1 \xmapsto{f}$

$0 \xmapsto{f}$

$1 \xmapsto{f}$

$2 \xmapsto{f}$

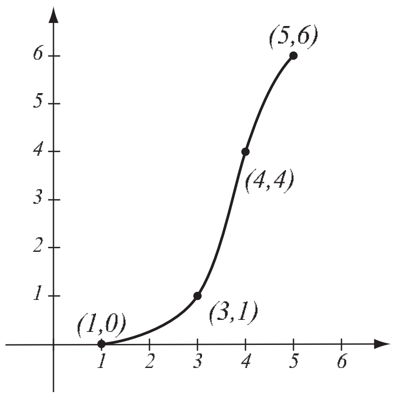
The reason $f(x) = x^2$ does not have an inverse is that it is not a one-to-one function. But then, what is $\sqrt{x}$?

We solve the problem of the function not being one-to-one by considering only one part of the graph to make it pass the horizontal line test. Another way to think of this is to make the domain smaller (we "restrict" the domain).

The function $\sqrt{x}$ is now the inverse of the new function with restricted domain.

**Note.** Only one-to-one functions have inverses.

**Example.** Let $f$ be given by the graph and the table. Write some values for $f^{-1}(x)$ and use them to help you sketch the graph of $f^{-1}(x)$.




| $x$ | $f(x)$ |
|---|---|
| 1 | 0 |
| 3 | 1 |
| 4 | 4 |
| 5 | 6 |

| $x$ | $f^{-1}(x)$ |
|---|---|
| | |

We see: the graph of $f^{-1}$ is obtained by reflecting the graph of $f$ in the line $y = x$.

Domain of $f$: Domain of $f^{-1}$:

Range of $f$: Range of $f^{-1}$:

The rule for domains and ranges of $f$ and $f^{-1}$ is:

Domain of $f$ = Range of $f^{-1}$ Range of $f$ = Domain of $f^{-1}$

**Example.** Draw the graphs of the functions and their inverses.

$y = x^2$

$y = x^3$

$y = 2 - x$

$y = \dfrac{1}{x}$

## 5.2 Exponential Functions

**Example.** Use the table to help you draw the graph of the function $f(x) = 2^x$.

| $x$ | $2^x$ |
|---|---|
| -2 | |
| -1 | |
| $-\frac{1}{2}$ | |
| 0 | |
| $\frac{1}{2}$ | |
| 1 | |
| 2 | |

**Example.** Use the table to help you draw the graph of the function $f(x) = \left(\frac{1}{2}\right)^x$.

| $x$ | $\left(\frac{1}{2}\right)^x$ |
|---|---|
| -2 | |
| -1 | |
| $-\frac{1}{2}$ | |
| 0 | |
| $\frac{1}{2}$ | |
| 1 | |
| 2 | |

**Note.** The graphs appear to be reflections of each other in the $y$-axis.

Verify:

Similarly we may consider a function of form $f(x) = b^x$, $b > 0$, $b \neq 1$.
Any such function is called an *exponential function*.

**Example.** Use the calculator to graph the functions $2^x$, $3^x$ on the same set of axes. Then graph the functions $\left(\frac{1}{2}\right)^x$, $\left(\frac{1}{3}\right)^x$.

Graphs of the exponential function $f(x) = b^x$, $b > 0$, $b \neq 1$.

**Compound interest formula**

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$A$ = amount in the account after $t$ years

$P$ = principal — amount deposited into the account

$r$ = annual interest rate as a decimal number

$t$ = time in years

$n$ = number of times per year interest is compounded

**Example.** $200 is deposited in an account bearing 5.5%, compounded monthly. How much is in the account in 30 months?

**Note.** If we set $b = \left(1 + \frac{r}{n}\right)^n$, the amount in the account after $t$ years is $A(t) = P \cdot b^t$.

| $t$ | $A(t)$ |
|---|---|
| 5 | |
| 10 | |
| 20 | |

**Example.** Consider the expression $\left(1 + \frac{1}{n}\right)^n$.

May think of this as the amount in the account if $P = 1$, $r = 100\%$, $t = 1$. Explore how this amount changes if we increase the number of compoundings in a year.

| $n$ | $\left(1 + \frac{1}{n}\right)^n$ |
|---|---|
| | |

The greater $n$ is, the closer $\left(1 + \frac{1}{n}\right)^n$ is to some number 2.718281 that we call $e$.

Like $\pi$, $e$ is a special number that crops up in many places in mathematics.
Like $\pi$, $e$ is irrational (infinite decimal, nonrepeating).

Often times the exponential function $e^x$ is called *the exponential function*.

## 5.3 Logarithmic Functions

Simply put, $\log_a x$ is the answer to the question $a^? = x$.

**Example.** Find the logarithms.

$\log_2 8 =$ $\qquad$ $\log_{10} 1000 =$ $\qquad$ $\log_4 16 =$

$\log_5 \frac{1}{5} =$ $\qquad$ $\log_8 \frac{1}{64} =$ $\qquad$ $\log_{\frac{1}{2}} 16 =$

$\log_7 \sqrt{7} =$ $\qquad$ $\log_5 \sqrt[3]{125}$ $\qquad$ $\log_2 \sqrt{\frac{1}{2}} =$

**Example.** Find the logarithms.

$\log_a 1 =$ $\qquad$ $\log_a a =$ $\qquad$ $\log_a a^4$

$\log_a \sqrt{a} =$ $\qquad$ $\log_a \sqrt[5]{a^4}$ $\qquad$ $\log_a \frac{1}{\sqrt[3]{a^7}} =$

**Definition.** $\log_a x$ is the number we raise $a$ to in order to get $x$. In other words,

$$y = \log_a x \text{ is the number such that } a^y = x$$

Using the last statement, we can always turn an exponential equation into a logarithmic one, and vice versa.

**Example.** Convert equation into other form logarithmic or exponential.

$14 = 2^x$ $\qquad$ $\log_b 4 = 7$

$a^3 = 17$ $\qquad$ $\log_2 y = 5$

**Special bases for logarithms.**

$a = 10$: instead of $\log_{10} x$ we write $\log x$

$a = e$: instead of $\log_e x$ we write $\ln x$ ("logarithm natural")

Thus, $\log \frac{1}{1000} =$ $\qquad$ $\ln \frac{1}{\sqrt{e}} =$

Calculators have logarithms with base 10 and $e$. To find logarithms for other bases using the calculator we use the

**Change of base formula:** $$\log_b M = \frac{\log_a M}{\log_a b}$$

**Example.**

$\log_2 3 =$

$\log_7 0.54 =$

**Exponential and logarithmic functions with same base are inverses of each other**

If $y = \log_a x$, then $x = a^y$. This means that we solved the equation $y = \log_a x$ for $x$, getting the inverse of the logarithmic function. We see that it is the exponential function with the same base. We can also say that the inverse of the exponential function is the logarithmic function.

We draw the graph of $y = \log_a x$ by thinking of it as the inverse of the function $a^x$.

If we let $f(x) = a^x$, then $f^{-1}(x) = \log_a x$. The usual identities for inverse functions take the form:

$f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = x$

$\log_a a^x = x$

$a^{\log_a x} = x$

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# 5.4 Properties of Logarithmic Functions

Recall the formulas:

$\log_a a = 1$ $\qquad$ $\log_a 1 = 0$

$\log_a a^x = x$ $\qquad$ $a^{\log_a x} = x$

**Example.** Simplify.

$10^{\log 5} =$ $\qquad$ $3^{\log_3 w^2} =$

$\log_7 7^{3x-1} =$ $\qquad$ $\ln e^{-t-3} =$

**More properties of logarithms.**

| Property | Related exponential property |
|---|---|
| $\log_a(MN) = \log_a M + \log_a N$ | $a^{x+y} = a^x \cdot a^y$ |
| $\log_a \dfrac{M}{N} = \log_a M - \log_a N$ | $a^{x-y} = \dfrac{a^x}{a^y}$ |
| $\log_a M^p = p \log_a M$ | $(a^x)^y = a^{xy}$ |

**Example.** Write as a sum and/or difference of logarithms. Express powers as factors. Simplify if possible.

$\log_4(16x) =$

$\log\left(\dfrac{y}{x^2}\right) =$

$\ln \dfrac{x^4}{\sqrt[3]{x+1}} =$

$\log_3 \sqrt[4]{9xy^2} =$

**Example.** Write as a single logarithm.

$2\log x + 3\log y =$

$\dfrac{1}{2}\log_6 z - 4\log_6 w =$

$\log_7 56 - \log_7 8 =$

$\ln(x^2 + 3x + 2) - 2\ln(x + 1) =$

# 5.5 Exponential and Logarithmic Equations

We consider some techniques for dealing with equations that involve exponential or logarithmic functions.

**Example.** Solve the equation by writing all expressions as powers of the same base.

$5^{2x+3} = \dfrac{1}{125}$

**Example.** How to deal with some exponential equations that cannot be written using the same base? Solve.

$8^{3x-4} = 5^x$

**Example.** Solve by recognizing quadratic form.

$e^{2x} - 3e^x - 40 = 0$

**Example.** Solve using properties of logarithms.

$\log(x-1)+\log(x+2)=1$

# 5.6 Exponential Functions Applications

**Example (Uninhibited population growth).** A model that gives the number of individuals (people, cells) after time $t$ is

$$P(t) = P_0 e^{kt},\ k > 0$$

where $P_0$ is the initial number of individuals, and $k$ is a positive constant that represents the growth rate ("exponential growth rate").

a) Suppose the exponential growth rate of a population of a city is 3%, and it starts with 89,000 people. Write the function describing the number $P(t)$ of people $t$ years after start. Graph the function.
b) How long until the population doubles?
c) Now suppose we have a city whose population in 1980 was 112,000 and in 2005 is 335,000. Write the function that describes the number of people living in the city $t$ years after 1980. What is the exponential growth rate?
d) When will the city from c) reach population 420,000?

**Example (Uninhibited radioactive decay).** The amount of radioactive material present at time $t$ is given by

$$P(t) = P_0 e^{-kt},\ k > 0$$

where $P_0$ is the original amount of radioactive material and $k$ is a positive constant, the decay rate. Usually, $k$ is expressed in terms of a half-life, which is the time required for half of the material to decay.

a) The half-life of carbon 14 (an unstable carbon) is 5730 years. Write the function that describes the amount of carbon as a function of $t$.
b) If the amount of carbon 14 is 10% of the original amount, how long has it been decaying?
c) In a living organism, the ratio of carbon 14 to carbon 12 is constant while the organism is alive. Once it dies, the amount or carbon 12 remains the same, while carbon-14 decays. If we know the ratio of carbon 14 to carbon 12 in a sample, this can help us determine when the organism died. Suppose the remains of a person found in an archaological site contain 20% of the original amount of carbon 14. How long ago did this person die?