Advanced Calculus 2 — Exam 1
MAT 526/626, Spring 2024 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when a function $f : A \to \mathbf{R}$ is strictly increasing.

Theory 2. (3pts) State the Bolzano Intermediate Value Theorem.

Theory 3. (3pts) State the theorem about uniform approximation of a continuous function by a polynomial.

Type A problems (5pts each)

A1. For every $a > 0$, show that the function $f(x) = \frac{1}{x}$ is Lipschitz on the interval $[a, \infty)$.

A2. Let (x_n, y_n) be a sequence of points in the plane that is in the set $A = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 4\}$. If (x_n) converges to c and (y_n) converges to d, show that $(c, d) \in A$.

A3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$ for a rational x, and $f(x) = x$ for an irrational x. Show that f is not continuous at every $c \neq 0, 1$.

A4. Show that $\sqrt{2}$ exists by showing the equation $x^2 = 2$ has a solution. Then find an interval of width $\frac{1}{16}$ that contains the solution.

A5. Show that the function $f(x) = \frac{1}{x}$ is not Lipschitz on $(0, 1]$ by using the sequential criterion to show that it is not uniformly continuous.

Type B problems (8pts each)

B1. Prove that the function $f(x) = x^3$ is continuous at every $c \in \mathbf{R}$.

B2. Let $f : [a, b] \to \mathbf{R}$ be a continuous function that takes on positive and negative values. Let $g(x) = f(x)^2$. Show that the range of g is $[0, M]$ for some number $M > 0$.

B3. For every $a > 0$, prove that the function $f(x) = \sqrt{x}$ is Lipschitz on $[a, \infty)$ and therefore continuous at every $c > 0$.

B4. Let $f : [a, b] \to \mathbf{R}$ be continuous. Suppose f takes on some value V_1 at least twice and there is no interval $[c, d] \subseteq [a, b]$ on which the function is constant. Show that there is another function value $V_2 \neq V_1$ that is taken on at least twice.

B5. Let $f : \mathbf{R} \to \mathbf{R}$ be an increasing function, and let $g(x) = f(x) + x$. Show that g is strictly increasing and is discontinuous at the same points where f is discontinuous.

B6. Suppose we have defined the function $f(x) = 2^x$ for all real numbers x and that it satisfies the usual algebraic properties of working with exponents, that it is continuous at 0, and that $2^x < 1$ for $x \leq 0$. Show that $f(x)$ is uniformly continuous on $(-\infty, 0]$.

Type C problems (12pts each)

C1. Recall the function defined below that is increasing and discontinuous at every rational number: $f(x) = \displaystyle\sum_{k\in\mathbf{N},\ q(k)\leq x} \frac{1}{2^k}$, where $q : \mathbf{N} \to \mathbf{Q}$ is a bijection.

Show that the function is continuous at every irrational number c using these two steps:

1) For any two numbers $u < x$, show that $f(x) - f(u) = \displaystyle\sum_{k\in\mathbf{N},\ u<q(k)\leq x} \frac{1}{2^k}$.

2) For any $M \in \mathbf{N}$ show there is a $\delta > 0$ such that $(c - \delta, c + \delta)$ does not contain any of the rational numbers $q(1), q(2), \ldots, q(M)$. Use this fact, along with 1) to argue that $f(x) - f(c)$ is small when x is near an irrational c.

C2. Let $f : (0, 1] \to \mathbf{R}$ be continuous, $f(x) > 0$ for all $x \in (0, 1]$ and let $\lim_{x\to 0+} f(x) = 0$. Show that the function $\frac{1}{f(x)}$ is not uniformly continuous. Probably best to use the sequential criterion. How to construct the sequences (x_n) and (u_n)?

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Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define the derivative of a function at a point.

Theory 2. (3pts) State your favorite L'Hospital's rule, taking care to clearly state the hypotheses and conclusion.

Theory 3. (3pts) State Carathéodory's Theorem.

Type A problems (5pts each)

A1. Is this function is differentiable at 0? $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

A2. Let f be differentiable at a, but not necessarily elsewhere. Show that

$$\lim_{h \to 0} \frac{f(a+3h) - f(a-2h)}{5h} = f'(a)$$

A3. Let $G(x)$ be a function such that $G'(x) = \sin(x^2)$ and $G(0) = 0$ (it exists, but cannot be written using elementary functions). Find $\displaystyle\lim_{x \to 0} \frac{G(2x)}{G(x)}$.

A4. Derive the product rule from the sum rule and the chain rule by taking the derivative of $\ln(f(x)g(x))$ in two ways: directly and after application of logarithmic rules.

A5. Use the Mean Value Theorem to show $|\sqrt{x} - \sqrt{y}| < \frac{1}{2}|x - y|$ for all $x, y \geq 1$.

A6. Show that $\left|\cos x - (1 - \frac{x^2}{2} + \frac{x^4}{24})\right| < \frac{1}{720}$ for all $|x| < 1$.

Type B problems (8pts each)

B1. Show that : a) f is differentiable at $x = 0$ b) f' is continuous on $\mathbf{R}$.

$$f(x) = \begin{cases} x^3 \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

B2. Prove the generalized Bernoulli's inequality: for every $x \geq -1$, $c \in \mathbf{Q}, c \geq 1$,

$$(1+x)^c \geq 1 + cx$$

To do this, consider the function $f(x) = (1+x)^c - 1 - cx$ on $[-1, \infty)$ and show that it has its absolute minimum at $x = 0$ (use a sign chart of f') and that $f(0) = 0$. How does this imply the inequality?

B3. Use a Taylor polynomial of degree 2 centered at $x_0 = \frac{9}{4}$ to get a rational number (you do not have to simplify it) that approximates $\sqrt{2}$. What is the accuracy of this approximation?

B4. Show that the equation $x - \cos x = 0$ has a solution in the interval $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and show that Newton's method converges regardless of the starting point in this interval.

B5. Let $f : [a, \infty) \to \mathbf{R}$ be differentiable on (a, ∞) and let $\lim_{x \to \infty} f'(x) = \infty$. Use the Mean Value Theorem to show $\lim_{x \to \infty} f(x) = \infty$.

B6. For $a, b \geq 0$, use convexity of $f(x) = -\sqrt[n]{x}$ (verify convexity first) to show that $\dfrac{\sqrt[n]{a} + \sqrt[n]{b}}{2} \leq \sqrt[n]{\dfrac{a+b}{2}}$.

Type C problems (12pts each)

C1. Show that, for any $n \in \mathbf{N}$, $\lim_{x \to 0^+} \dfrac{e^{-\frac{1}{x}}}{x^n} = 0$. *(Hint: requires a trick.)*

Advanced Calculus 2 — Exam 3 **MAT 526/626, Spring 2024 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Assuming tagged partitions have been defined, define the Riemann integral of a function $f : [a, b] \to \mathbf{R}$.

Theory 2. (3pts) State Cauchy's criterion of integrability.

Theory 3. (3pts) State the first form of the Fundamental Theorem of Calculus (the one dealing with how to compute the integral of a function using an antiderivative).

Type A problems (5pts each)

A1. Give an example of a function $f : [0, 1] \to \mathbf{R}$ that is Riemann-integrable on $[0, 1]$ and $\int_c^d f = 0$ for every interval $[c, d] \subseteq [0, 1]$, but f is not the constant zero function. Justify.

A2. Let $f : [0, 1] \to \mathbf{R}$ be the function at right. Use Cauchy's criterion to show f is not Riemann-integrable.

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbf{Q} \\ 0, & \text{if } x \notin \mathbf{Q} \end{cases}$$

A3. If $F(x) = \displaystyle\int_{\cos x}^{\ln x} \tan(t^2 + t)\, dt$, find the expression for $F'(x)$. Justify.

A4. For a function $g : [a, b] \to \mathbf{R}$, suppose there exist sequences of functions (f_n) and (h_n) and real numbers (c_n) such that $f_n(x) \le g(x) \le h_n(x)$ and $h_n(x) - f_n(x) < c_n$ for every $x \in [a, b]$, and that $\lim c_n = 0$. Show that g is Riemann-integrable on $[a, b]$.

A5. Let $f : [a, b] \to \mathbf{R}$ be continuous, $f(x) \ge 0$ for every $x \in [a, b]$ and $\int_a^b f = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

A6. Write the specific expression (i.e., with numbers, not variables) for the trapezoid estimate T_4 of the integral $\int_1^3 \ln x\, dx$, but do not evaluate it. Determine its accuracy.

Type B problems (8pts each)

B1. Let $f : [1, \infty] \to \mathbf{R}$ be the function at right and let $F : [1, \infty) \to \mathbf{R}$, $F(x) = \int_1^x f$.
a) Calculate $F(x)$.
b) Draw the graphs of f and F.
c) Where is F continuous? Differentiable?

$$f(x) = \begin{cases} 2, & \text{if } x \in [1, 3] \\ -1, & \text{if } x \in (3, 6) \\ 7 - x, & \text{if } x \in [6, \infty) \end{cases}$$

B2. Let $f : [1, 3] \to \mathbf{R}$ be the function at right.
a) Guess the value of $\int_1^3 f$.
b) Prove by definition of the Riemann integral that $\int_1^3 f$ is the number you guessed.

$$f(x) = \begin{cases} -2, & \text{if } x \in [1, 2) \\ 3, & \text{if } x \in [2, 3] \end{cases}$$

B3. Let $f \in \mathcal{R}[a,b]$ be continuous at $c \in [a,b]$. Show that the indefinite integral $F(x) = \int_a^x f$ is differentiable at c and $F'(c) = f(c)$.

B4. Let $f : [a,b] \to \mathbf{R}$ be a bounded function such that for every $c > a$, the restriction $f|_{[c,b]}$ is Riemann-integrable. Use the squeeze theorem to show that

1) f is Riemann-integrable on $[a,b]$.
2) $\int_a^b f = \lim_{c\to a+} \int_c^b f$.

B5. If $f : \mathbf{R} \to \mathbf{R}$ is continuous and has the property $\int_0^{2x} f = 2\int_0^x f$ for all $x \in \mathbf{R}$, show that $f(x)$ is constant. *Hint: Show first that $f(2x) = f(x)$ for every $x \in \mathbf{R}$, then for any x consider* $\lim_n f\left(\frac{x}{2^n}\right)$.

B6. For the integral $\int_{0.01}^1 \sin\frac{1}{x}\,dx$, how many subintervals are needed so that the midpoint estimate M_n has accuracy 10^{-3}?

Type C problems (12pts each)

C1. If p is a polynomial of degree at most 3, show the Simpson approximation S_n is exact as follows, without using the error estimate. First note that it is enough to show this for the case of two subintervals (i.e., for S_2).
a) For each of the functions $f(x) = 1, x, x^2, x^3$ show that

$$\int_{a-h}^{a+h} f(x)\,dx = \frac{1}{3}h(f(a-h) + 4f(a) + f(a+h)).$$

b) Conclude that if p is any polynomial of degree at most 3, $x_0 < x_1 < x_2$, $x_1 = \frac{x_0+x_2}{2}$, and $h = x_1 - x_0$, then

$$\int_{x_0}^{x_2} p(x)\,dx = \frac{1}{3}h(p(x_0) + 4p(x_1) + p(x_2)),$$

which proves the Simpson approximation is exact for S_2.
c) Conclude the Simpson approximation S_n is exact for any polynomial p of degree at most 3, and any even n.