Advanced Calculus 2 — Exam 3
MAT 526/626, Spring 2020 — D. Ivanšić

Name: ____________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define the indefinite integral based at point a.

Theory 2. (3pts) State the squeeze theorem (criterion of integrability).

Theory 3. (3pts) State the first form of the Fundamental Theorem of Calculus (the one dealing with how to compute the integral of a function using an antiderivative).

Type A problems (5pts each)

A1. Give an example of a function that is not Riemann-integrable. Show, using the definition, that it is not Riemann-integrable.

A2. Prove the Mean Value Theorem for Integrals: if $f(x)$ is continuous on $[a, b]$, there exists a $c \in [a, b]$ such that $\int_a^b f = f(c)(b-a)$.

A3. If $F(x) = \displaystyle\int_{\sqrt{x}}^{e^x} \sin \frac{1}{t^2}\, dt$, find the expression for $F'(x)$.

A4. For a function $f : [a, b] \to \mathbf{R}$, suppose there exist sequences of tagged partitions $\dot{\mathcal{P}}_n$, $\dot{\mathcal{Q}}_n$ such that $||\dot{\mathcal{P}}_n|| \to 0$ and $||\dot{\mathcal{Q}}_n|| \to 0$, and $|S(f, \dot{\mathcal{P}}_n) - S(f, \dot{\mathcal{Q}}_n)| \geq \frac{1}{10}$ for all $n \in \mathbf{N}$. Show that f is not Riemann-integrable on $[a, b]$.

A5. If $f : [0, 1] \to \mathbf{R}$ is continuous and has the property $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

Type B problems (8pts each)

B1. Let $f : [-2, \infty] \to \mathbf{R}$ be the function at right and let $F : [-2, \infty) \to \mathbf{R}$, $F(x) = \int_{-2}^x f$.
a) Calculate $F(x)$.
b) Draw the graphs of f and F.
c) Where is F continuous? Differentiable?

$$f(x) = \begin{cases} 3, & \text{if } x \in [-2, 1] \\ 2 + x^2, & \text{if } x \in (1, 2) \\ x, & \text{if } x \in [2, \infty) \end{cases}$$

B2. Let $f : [2, 6] \to \mathbf{R}$ be the function at right.
a) Guess the value of $\int_2^6 f$.
b) Prove by definition of the Riemann integral that $\int_2^6 f$ is the number you guessed.

$$f(x) = \begin{cases} -1, & \text{if } x \in [2, 3) \\ 2, & \text{if } x \in [3, 6] \end{cases}$$

B3. Let $f : [0,1] \to \mathbf{R}$ be the function at right. Use the squeeze theorem to show f is Riemann-integrable on $[0,1]$.

$$f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \in (0,1] \\ 0, & \text{if } x = 0 \end{cases}$$

B4. Let $f : [a,b] \to \mathbf{R}$ be an increasing function. Use the squeeze theorem to show f is Riemann integrable on $[a,b]$. (Hint: subdivide the interval $[a,b]$ into n equal subintervals and define functions α and ω in an easy way on each subinterval to ensure $\alpha \le f \le \omega$.)

B5. Let $f \in \mathcal{R}[0,a]$ and let $g : [0,a] \to \mathbf{R}$, $g(x) = f(a-x)$. Use Cauchy's criterion to show that $g \in \mathcal{R}[0,a]$. (Hint: relate tagged partitions and Riemann sums for g to those for f.)

Type C problems (12pts each)

C1. Thomae's function $f : [0,1] \to \mathbf{R}$ is given by

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ (reduced) for some } m, n \in \mathbf{N} \\ 0, & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Use the definition to show that f is Riemann-integrable on $[0,1]$, following these steps:
a) For $i \ge 2$, show there are at most $i-1$ reduced fractions in $[0,1]$ with denominator i.
b) Given $n \in \mathbf{N}$, let D_n be the set of reduced fractions in $[0,1]$ with denominators $1, 2, \ldots, n$. Show D_n has at most $2 + 1 + 2 + 3 + \cdots + (n-1) = 2 + \frac{n(n-1)}{2}$ elements.
c) Observe that $2 + \frac{n(n-1)}{2} \le \frac{n^2}{2}$ for $n \ge 4$.
d) Let $\dot{\mathcal{P}}_n$ be a tagged partition with n^3 equal subintervals and any tags. Show at most n^2 of those subintervals contain an element of D_n — the remaining ones do not have any elements of D_n (note an element of D_n may be an endpoint of a subinterval, putting it in two subintervals).
e) Use d) to argue that $S(f, \dot{\mathcal{P}}_n) < \frac{2}{n}$. Conclude f is Riemann-integrable.