| **Advanced Calculus 2 — Exam 2** |
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| **MAT 526/626, Spring 2020 — D. Ivanšić** |

**Name:** ______________________________

*Show all your work!*

*Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define when a function $f : I \to \mathbf{R}$ has a relative maximum at a number $c$.

**Theory 2.** (3pts) Define the $n$-th Taylor polynomial at $x_0$ for a function $f$.

**Theory 3.** (3pts) State the Cauchy Mean Value Theorem.

## Type A problems (5pts each)

**A1.** Show this function is differentiable at 0 and find $f'(0)$: $f(x) = \begin{cases} \sin x, & \text{if } x > 0 \\ e^x - 1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$

**A2.** Find the limit: $\lim\limits_{x\to 0+} \ln(x+1)\ln x$.

**A3.** Derive the quotient rule from the product rule as follows: let $h(x) = \frac{f(x)}{g(x)}$, then $h(x)g(x) = f(x)$. Assume $h$ is differentiable and get $h'$ from the second equation after applying the product rule.

**A4.** Use the Mean Value Theorem to show $|\tan x - \tan y| \geq |x - y|$ for all $x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

**A5.** Let $f(x) = \sqrt[3]{1+x}$.
a) Write the Taylor polynomials $P_1$ and $P_2$ for $f$ at $x_0 = 0$.
b) Show that for $x > 0$, $P_2(x) \leq \sqrt[3]{1+x} \leq P_1(x)$.

**A6.** Let $I$ be an open interval, $a \in I$, and let $f : I \to \mathbf{R}$ be differentiable. Show: if $\lim\limits_{x\to a} f'(x)$ exists, it is equal to $f'(a)$.

## Type B problems (8pts each)

**B1.** Let $f : [-1, 1] \to \mathbf{R}$ be the piecewise-linear function, where, for $n = 2, 3, \ldots$

$$f(x) = \begin{cases} \text{line between points } \left(\frac{1}{n}, \frac{1}{n^2}\right) \text{ and } \left(\frac{1}{n-1}, \frac{1}{(n-1)^2}\right) \text{ if } x \in \left(\frac{1}{n}, \frac{1}{n-1}\right] \\ \text{line between points } \left(-\frac{1}{n-1}, \frac{1}{(n-1)^2}\right) \text{ and } \left(-\frac{1}{n}, \frac{1}{n^2}\right) \text{ if } x \in \left[-\frac{1}{n-1}, -\frac{1}{n}\right) \\ 0, \text{ if } x = 0. \end{cases}$$

Show that $f$ is differentiable at 0. (Best to use an $\epsilon$-$\delta$ argument and inequality involving the function.)

**B2.** Let $f : [-1, 4] \to \mathbf{R}$ be continuous on $[-1, 4]$ and differentiable on $(-1, 4)$, and suppose that $-2 \le f'(x) \le 7$ for all $x \in (-1, 4)$. If $f(-1) = 21$, use the Mean Value Theorem to establish the range of values that $f(4)$ can take. Give examples to show that the upper and lower bound for $f(4)$ can be achieved.

**B3.** Let $f(x) = x^2 \sin \dfrac{1}{x}$ and $g(x) = x$. Show that $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)}$ exists, but $\lim\limits_{x \to 0} \dfrac{f'(x)}{g'(x)}$ does not. Does this contradict L'Hospital's rule?

**B4.** Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that $f'''$ is continuous and for some $c \in \mathbf{R}$, $f'(c) = f''(c) = 0$ and $f'''(c) > 0$. Use Taylor's theorem to show that $f$ does not have a local extreme at $c$.

**B5.** Use a Taylor polynomial to get a rational number (you do not have to simplify it) that approximates $\cos \frac{2}{3}$ with accuracy $10^{-3}$.

**B6.** Show that the equation $x^3 + 3x^2 + 7x + 2 = 0$ has a solution and find an interval in which Newton's method converges regardless of the starting point.

## Type C problems (12pts each)

**C1.** Let $I$ be an open interval and let $f : I \to \mathbf{R}$ be differentiable and convex. Note we are not assuming that $f''$ exists. Show that $f'$ is an increasing function as follows: let $a, b \in I$, $a < b$ and let $x_t = (1-t)a + tb$. Then $f'(a) = \lim\limits_{x_t \to a+} \dfrac{f(x_t) - f(a)}{x_t - a}$. Turn the limit into a limit by $t$ and apply convexity to get an inequality that will help you show $f'(a) < f'(b)$.