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| **Advanced Calculus 2 — Exam 1**<br>**MAT 526/626, Spring 2020 — D. Ivanšić** | **Name:** ______________________<br>*Show all your work!* |

*Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define when a function $f : A \to \mathbf{R}$ is continuous at a point $c \in A$.

**Theory 2.** (3pts) State the Maximum-Minimum Theorem.

**Theory 3.** (3pts) State the theorem that says what is the image of $[a, b]$ under a continuous function.

## Type A problems (5pts each)

**A1.** Prove by definition that the function $f(x) = 7x + 2$ is continuous at every $c \in \mathbf{R}$.

**A2.** Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and $A = \{x \in \mathbf{R} \mid f(x)^2 - f^2(x) = 3\}$ (the first one is $f \cdot f$, the second one is $f \circ f$). If $(x_n)$ is a sequence such that $x_n \in A$ and $(x_n)$ converges to $c$, show that $c \in A$.

**A3.** Let $f, g : \mathbf{R} \to \mathbf{R}$ be functions so that $f$ is continuous at a point $c \in \mathbf{R}$ and $g$ is not continuous at $c$. Show that $f + g$ is not continuous at $c$.

**A4.** Prove that the equation $2^x = \sin x$ has infinitely many solutions. (Draw a picture for inspiration.)

**A5.** Use the sequential criterion to show that $f(x) = x^2$ is not uniformly continuous on its domain $\mathbf{R}$.

## Type B problems (8pts each)

**B1.** Prove by definition that the function $f(x) = x^2 - 5x + 3$ is continuous at every $c \in \mathbf{R}$.

**B2.** Let $f : (0, \infty) \to \mathbf{R}$ be defined by $f(x) = x^2$ for a rational $x$, and $f(x) = 3$ for an irrational $x$. Determine the numbers where the function is continuous and where it is not. Justify in detail. You may use the fact that $x^2$ is a continuous function on all reals.

**B3.** Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists a $c \in [0, 1]$ such that $f(c) = c$. (To get started, draw a picture.)

**B4.** Let $f, g : [a, b] \to \mathbf{R}$ be Lipschitz functions. Show that the function $f \cdot g$ is Lipschitz.

**B5.** Let $f : [a, b] \to \mathbf{R}$ be continuous and nonconstant, and suppose $f(a) = f(b)$. Show that there is a function value $V \neq f(a), f(b)$ that is taken on at least twice.

**B6.** For $x > 0$, recall that $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$, $m \in \mathbf{Z}$, $n \in \mathbf{N}$, where the first equation is the definition and the second was proven (so you may use it). Show that rational exponents are well-defined, that is, if $mq = np$, show that $(x^{\frac{1}{n}})^m = (x^{\frac{1}{q}})^p$. Start with $x^{mq} = x^{np}$, raise both sides to $\frac{1}{q}$, and take it from there. This is not hard, but be careful that you not use what you are trying to prove.

## Type C problems (12pts each)

**C1.** Let $f : \mathbf{Q} \to \mathbf{R}$ be a function that, for every $c \in \mathbf{R}$, is uniformly continuous on an interval around $c \in \mathbf{R}$. That is, for every $c \in \mathbf{R}$, there is an interval $(c - \delta_c, c + \delta_c)$ such that the restriction of $f$ to $(c - \delta_c, c + \delta_c) \cap \mathbf{Q}$ is uniformly continuous.
a) Let $c \in \mathbf{R}$ and let $x_n \to c$, where $x_n \in \mathbf{Q}$ for every $n \in \mathbf{N}$. Show that $f(x_n)$ is a Cauchy sequence, hence converges.
b) For $c \in \mathbf{R} - \mathbf{Q}$, define $f(c) = \lim f(x_n)$, where $x_n \in \mathbf{Q}$ is any sequence that converges to $c$. Show that this definition does not depend on the sequence $x_n$, that is, if $x_n, y_n \in \mathbf{Q}$, $x_n \to c$, $y_n \to c$, then $\lim f(x_n) = \lim f(y_n)$.
This allows us to extend the function $f : \mathbf{Q} \to \mathbf{R}$ to all real numbers.

**C2.** Show that the extension of the function $f : \mathbf{Q} \to \mathbf{R}$, as defined above, is uniformly continuous on an interval around every $c \in \mathbf{R}$, and is thus continuous.