7.1 Riemann Integral

We wish to precisely define $\int_a^b f(x)dx$.

Definition. Let $I = [a, b]$. A *partition* of I is a finite ordered set $\mathcal{P} = (x_0, x_1, \dots, x_n)$ where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The points of $\mathcal{P}$ divide I into subintervals $[x_0, x_1], [x_1, x_2], \dots [x_{n-1}, x_n]$ of lengths $\Delta x_1, \Delta x_2, \dots \Delta x_n$, where $\Delta x_i = x_i - x_{i-1}$.

The *norm (mesh)* of a partition is $||\mathcal{P}|| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$ (biggest subinterval length).

If we choose numbers $t_i \in [x_{i-1}, x_i]$, $i = 1, \dots, n$, then the t_i's are called *tags* of the subintervals $[x_{i-1}, x_i]$. A *tagged partition* $\dot{\mathcal{P}}$ is a partition $\mathcal{P}$ along with a set of tags $t_i \in [x_{i-1}, x_i]$, $i = 1, \dots, n$. (Formally: $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i) \mid x_i \in \mathcal{P}, i = 1, \dots, n\}$.) We define the norm a tagged partition as $||\dot{\mathcal{P}}|| = ||\mathcal{P}||$.

Definition. The *Riemann sum* of a function $f : [a, b] \to \mathbf{R}$ corresponding to the tagged partition $\dot{\mathcal{P}}$ is

$$S(f, \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)\Delta x_i.$$

(Use same notation if $\dot{\mathcal{P}}$ is a subpartition.)

Definition 7.1.1. A function $f : [a, b] \to \mathbf{R}$ is *Riemann integrable* on $[a, b]$ if there exists a number $L \in \mathbf{R}$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ s.t. if $\dot{\mathcal{P}}$ is any tagged partition of $[a, b]$ with $||\dot{\mathcal{P}}|| < \delta$, then $|S(f, \dot{\mathcal{P}}) - L| < \epsilon$.

Definition. If a function f is Riemann integrable, the number L above is denoted $\int_a^b f$ or $\int_a^b f(x)\,dx$, and called the *Riemann integral* of f over $[a,b]$.

$\mathcal{R}[a,b]$= the set of all Riemann-integrable functions on $[a,b]$

Theorem 7.1.2 If f is Riemann-integrable over $[a,b]$, then the value of the integral is uniquely determined.

Proof.

Example. If $f(x)=k$, then $f\in\mathcal{R}[a,b]$ and $\int_a^b f = k(b-a)$.

Example. Let $f(x)=\begin{cases} 2, & \text{if } x\in[0,1] \\ 5, & \text{if } x\in(1,3]. \end{cases}$ Then f is Riemann-integrable and $\int_0^3 f = 12$.

Example. $\int_0^4 x\,dx = 8$

Example. Let $E \subset [a, b]$ be a set of k elements and define $f(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \in [a, b] - E. \end{cases}$
Then $\int_a^b f = 0$.

Theorem 7.1.5. Let $f, g \in \mathcal{R}[a, b]$. Then

1) If $k \in \mathbf{R}$, then $kf \in \mathcal{R}[a, b]$ and $\int_a^b kf = k \int_a^b f$.
2) $f + g \in \mathcal{R}[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
3) If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

Proof.

Theorem 7.1.6. If $f \in \mathcal{R}[a, b]$, then f is bounded on $[a, b]$.

Proof.

7.2 Riemann-Integrable Functions

Cauchy Criterion 7.2.1. The function $f : [a, b] \to \mathbf{R}$ is in $\mathcal{R}[a, b]$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any tagged partitions of $[a, b]$ with $\min\{||\dot{\mathcal{P}}||, ||\dot{\mathcal{Q}}||\} < \delta$ then $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \epsilon$.

Proof.

Example. The Cauchy criterion is often useful to prove that a function is not Riemann-integrable.

Note: $f \notin \mathcal{R}[a,b] \iff$ there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ there exist tagged partitions $\dot{\mathcal{P}}$, $\dot{\mathcal{Q}}$ with $||\dot{\mathcal{P}}||, ||\dot{\mathcal{Q}}|| < \delta$ and $|S(f,\dot{\mathcal{P}}) - S(f,\dot{\mathcal{Q}})| \geq \epsilon_0$

Let $f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q} \\ 0, & \text{if } x \notin \mathbf{Q}. \end{cases}$ This is not a Riemann-integrable function.

Squeeze Theorem 7.2.3. $f \in \mathcal{R}[a,b]$ if and only if for every $\epsilon > 0$ there exist functions $\alpha, \omega \in \mathcal{R}[a,b]$ such that $\alpha(x) \leq f(x) \leq \omega(x)$ on $[a,b]$ and $\int_a^b \omega - \alpha < \epsilon$.

Proof.

Lemma 7.2.4. Let $J \subseteq [a,b]$ be an interval with endpoints $c \leq d$ and set

$$\phi(x) = \begin{cases} 1, & \text{if } x \in J \\ 0, & \text{if } x \in [a,b] - J. \end{cases}$$

Then $\phi \in \mathcal{R}[a,b]$ and $\int_a^b \phi = d - c$. (Note one or both endpoints need not be included in J.)

Recall Definition 5.4.9: a function $\phi : [a, b] \to \mathbf{R}$ is called a *step function* if there exists a collection of disjoint intervals $J_1, \ldots J_n$ (open, closed or half-open) such that $\bigcup_{k=1}^{n} J_k = [a, b]$ and s is constant on J_k, $k = 1, \ldots n$. If $\phi_1, \ldots, \phi_n$ are functions like in above lemma (ϕ_k is 1 on J_k, elsewhere 0), then $\phi = \sum_{j=1}^{n} k_j \phi_j$. Note that step functions only have finitely many values.

Theorem 7.2.5. Any step function $\phi : [a, b] \to \mathbf{R}$ is Riemann-integrable.

Proof. It is a linear combination of integrable functions.

Theorem 7.2.7. If $f : [a, b] \to \mathbf{R}$ is continuous, then $f \in \mathcal{R}[a, b]$.

Proof.

Theorem 7.2.8. If $f : [a, b] \to \mathbf{R}$ is monotone, then $f \in \mathcal{R}[a, b]$.

Proof. Essentially same idea as above.

Additivity Theorem 7.2.9. Let $f : [a, b] \to \mathbf{R}$, $c \in (a, b)$. Then $f \in \mathcal{R}[a, b]$ if and only if the restrictions of f to $[a, c]$ and $[c, b]$ are both Riemann-integrable. In that case, $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof.

Corollary 7.2.10. If $f \in \mathcal{R}[a, b]$ and $[c, d] \subseteq [a, b]$, then the restriction $f|_{[c,d]} \in \mathcal{R}[c, d]$.

Proof.

Definition 7.2.12. If $f \in \mathcal{R}[a, b]$ and $\alpha, \beta \in [a, b]$ with $\alpha < \beta$, we define $\int_\beta^\alpha f = -\int_a^\beta f$ and $\int_\alpha^\alpha f = 0$.

Theorem 7.2.13. Let $f : [a, b] \to \mathbf{R}$, $\alpha, \beta, \gamma \in (a, b)$. Then $\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$.

Proof.

7.3 The Fundamental Theorem

The Fundamental Theorem of Calculus (First form) 7.3.1. Suppose $E \subset [a, b]$ is a finite set, and $f, F : [a, b] \to \mathbf{R}$ are functions with properties:

1) F is continuous on $[a, b]$.
2) $F'(x) = f(x)$ for all $x \in [a, b] - E$.
3) $f \in \mathcal{R}[a, b]$.

$$\text{Then } \int_a^b f = F(b) - F(a).$$

Proof.

Example. Compute $\int_{-3}^{5} f$ using the fundamental theorem of calculus, if $f : \mathbf{R} \to \mathbf{R}$ is given by ($f(x) = \text{sign}(x)$)

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Example. Let $F : \mathbf{R} \to \mathbf{R}$ be given by

$$F(x) = \begin{cases} x^2 \cos \dfrac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Definition 7.3.3. If $f \in \mathcal{R}[a, b]$, we can define

$$F(x) = \int_a^x f, \text{ for } x \in [a, b].$$

F is called the *indefinite integral of f with basepoint a.*

Theorem 7.3.4. The indefinite integral F is continuous on $[a, b]$.
Furthermore, if $|f(x)| \leq M$ on $[a, b]$, then $|F(x) - F(u)| \leq M|x - u|$ for all $x, u \in [a, b]$, that is, the function F is Lipschitz (and therefore continuous).

Proof.

The Fundamental Theorem of Calculus (Second form) 7.3.5. If $f \in \mathcal{R}[a, b]$ is continuous at $c \in [a, b]$, then the indefinite integral F is differentiable at c and $F'(c) = f(c)$.

Proof.

Theorem 7.3.6. If f is continuous on $[a, b]$, then the indefinite integral F is differentiable on $[a, b]$ and $F' = f$.

Example. If $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = \operatorname{sign}(x)$, find the indefinite integral with basepoint -2.

Substitution Theorem 7.3.8. Let $J = [\alpha, \beta]$, and let $\phi : J \to \mathbf{R}$ be a function that has a continuous derivative. If $f : I \to \mathbf{R}$ is continuous and $\phi(J) \subseteq I$, then

$$\int_{\alpha}^{\beta} f(\phi(t))\phi'(t)\,dt = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx$$

Proof.

Example. Note that ϕ need not be injective.

$$\int_0^{\frac{5\pi}{2}} \frac{\cos t}{2 + \sin t}\,dt =$$

Example. Can you use substitution on this integral?

$$\int_0^4 \frac{\sin \sqrt{t}}{\sqrt{t}}\,dt =$$

Definition 7.3.10.

a) A set $Z \subseteq \mathbf{R}$ is a *null set* (not same as empty set) if for every $\epsilon > 0$ there exists a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals so that

$$Z \subseteq \bigcup_{k=1}^{\infty}(a_k, b_k) \text{ and } \sum_{k=1}^{\infty}(b_k - a_k) < \epsilon.$$

b) A statement about an $x \in I$ is said to hold *almost everywhere* (a.e.) if it holds for all $x \in I - Z$, where $Z \subseteq I$ is a null set.

Example. $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is a null set.

Example. $\mathbf{Q}_1 = \mathbf{Q} \cap [0, 1]$ is a null set.

Lebesgue's Integrability Criterion 7.3.12. A bounded function $f : [a, b] \to \mathbf{R}$ is Riemann-integrable if and only if it is continuous almost everywhere on $[a, b]$.

Example. $f : [0, 1] \to \mathbf{R}$

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q} \\ 0, & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Example. $f : [0, 1] \to \mathbf{R}$ given by:

$$f(x) = \begin{cases} 1, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbf{N} \\ 0, & \text{otherwise.} \end{cases}$$

Example. Thomae's function $f : [0, 1] \to \mathbf{R}$.

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ for some } m, n \in \mathbf{N} \\ 0, & \text{if } x \notin \mathbf{Q}. \end{cases}$$

The Composition Theorem 7.3.14. Let $f \in \mathcal{R}[a, b]$, $f([a, b]) \subseteq [c, d]$ and let $\phi : [c, d] \to \mathbf{R}$ be a continuous function. Then $\phi \circ f \in \mathcal{R}[a, b]$.

Proof.

Corollary 7.3.15. Let $f \in \mathcal{R}[a, b]$, and $|f| \leq M$ on $[a, b]$. Then $|f| \in \mathcal{R}[a, b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f| \leq M(b - a).$$

Proof.

The Product Theorem 7.3.16. If $f, g \in \mathcal{R}[a, b]$, then $f \cdot g \in \mathcal{R}[a, b]$.

Proof.

Integration by Parts 7.3.17. Let $F, G : [a, b] \to \mathbf{R}$ be differentiable on $[a, b]$, $f = F'$, $g = G'$, so that $f, g \in \mathcal{R}[a, b]$. Then

$$\int_a^b fG = FG\Big|_a^b - \int_a^b Fg$$

Proof.

Taylor's theorem with remainder in integral form 7.3.18. Let $f : [a, b] \to \mathbf{R}$ and suppose $f', f'', \ldots, f^{(n+1)}$ exist, and $f^{(n+1)} \in \mathcal{R}[a, b]$. Then

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n,$$

$$\text{where } R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,dt.$$

Proof.

7.5 Approximate Integration

Example. Evaluate $\int_0^1 \sin x^2\, dx$.

Using the fundamental theorem of calculus, we'd have to find the antiderivative of $\sin x^2$. Unfortunately, it can be shown that this antiderivative is not expressible using elementary functions, so we will resort to approximating the integral using Riemann sums.

Note. The fact that $\sin x^2$ does not have an elementary antiderivative is not surprising or profound, but merely the result of what functions we call "elementary." If our world of functions was only rational functions, many would not have an antiderivative in this world. For example, antiderivatives of $\frac{1}{x}$ or $\frac{1}{1+x^2}$ are $\ln x$ and $\arctan x$, which are not rational functions.

We consider Riemann sums with equal-length subintervals and consistently choose the sample points to be left, right or midpoints of the subintervals.

Definition. Set $h = \dfrac{b-a}{n}$. We give the following names to certain Riemann sums $\displaystyle\sum_{i=1}^{n} f(t_i)h$:

Left sum (approximation) L_n: uses $t_i = x_{i-1}$ (left endpoint of each subinterval)
Right sum (approximation) R_n: uses $t_i = x_i$ (right endpoint of each subinterval)
Midpoint sum (approximation) M_n: uses $t_i = \dfrac{x_{i-1}+x_i}{2}$ (midpoint of each subinterval)
Trapezoid sum (approximation) $T_n = \dfrac{L_n+R_n}{2}$: not a Riemann sum, but sum of areas of approximating trapezoids.

Note. The midpoint rectangle has the same area as a "tangent trapezoid" at the midpoint.

How good are these estimates? Let's start with an increasing function f:

Theorem 7.5.1. If $f : [a, b] \to \mathbf{R}$ is monotone, then

$$\left| \int_a^b f - T_n \right| \leq |f(b) - f(a)| \frac{b-a}{2n}$$

Example. Use this estimate on $\int_0^1 \sin x^2 \, dx$ to see how many subintervals are needed for accuracy 10^{-5}.

Theorems 7.5.3, 7.5.4, 7.5.6, 7.5.7. Let $f : [a, b] \to \mathbf{R}$ be such that f'' is continuous on $[a, b]$ and let B_2 be such that $|f''(x)| \leq B_2$ for all $x \in [a, b]$. Then there exist points $c, d \in [a, b]$ such that

$$T_n - \int_a^b f = \frac{(b-a)h^2}{12} \cdot f''(c) \text{ and } \int_a^b f - M_n = \frac{(b-a)h^2}{24} \cdot f''(d)$$

which implies

$$\left| T_n - \int_a^b f \right| \leq \frac{B_2(b-a)h^2}{12} = \frac{B_2(b-a)^3}{12n^2} \text{ and } \left| M_n - \int_a^b f \right| \leq \frac{B_2(b-a)h^2}{24} = \frac{B_2(b-a)^3}{24n^2}$$

Proof of midpoint formula.

Example. Use the trapezoid estimate on $\int_0^1 \sin x^2 \, dx$ to see how many subintervals are needed for accuracy 10^{-5}.

Simpson's Rule. Presumably we will get better accuracy if we approximate curvy areas with curvy objects such as parabolas.

The *Simpson approximation* is given by (n even):

$$S_n = \frac{1}{3}h(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

Theorems 7.5.8, 7.5.9. Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(4)}$ is continuous on $[a, b]$ and let B_4 be such that $|f^{(4)}(x)| \le B_4$ for all $x \in [a, b]$. Then there exists a point $c \in [a, b]$ such that

$$S_n - \int_a^b f = \frac{(b-a)h^4}{180} \cdot f^{(4)}(c), \text{ implying } \left| S_n - \int_a^b f \right| \le \frac{B_4(b-a)h^4}{180} = \frac{B_4(b-a)^5}{180n^4}$$

Proof omitted.

Example. Use the Simpson's rule estimate on $\int_0^1 \sin x^2 \, dx$ to see how many subintervals are needed for accuracy 10^{-5}.