Advanced Calculus 2 — Lecture notes
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6.1 The Derivative

Definition 6.1.1. Let $I \subseteq \mathbf{R}$, be an interval, $f : I \to \mathbf{R}$ a function and let $c \in I$. We say that L is the *derivative of f at c* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in I$, then $\left|\frac{f(x) - f(c)}{x - c} - L\right| < \varepsilon$. If L exists, we say f is differentiable at c and write $L = f'(c)$. In other words

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}, \text{if the limit exists.}$$

We can form a function $f' : J \to \mathbf{R}$, where $J \subseteq I$ is all points x for which $f'(x)$ exists, and define $f'(x) =$ the derivative of f at x.

Note. The point c could be the endpoint of the interval, in which case the limit above is one-sided.

Example. The constant function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = b$, is differentiable at every point, and $f'(x) = 0$.

Example. The identity function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x$, is differentiable at every point, and $f'(x) = 1$.

Example. The absolute value function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = |x|$, is differentiable at every $x \neq 0$, and is not differentiable at 0.

Theorem 6.1.2. If $f : I \to \mathbf{R}$ has a derivative at $c \in I$, then f is continuous at c.

Proof.

Note. A function that is continuous at c need not have a derivative at c, for example $|x|$ and $\sqrt{x}$ are not differentiable at 0.

Theorem 6.1.3 Differentiation Rules. Let $f, g : I \to \mathbf{R}$ be differentiable at $c \in I$, $\alpha \in \mathbf{R}$. Then the functions αf, $f \pm g$, fg and $\frac{f}{g}$ are differentiable at c and:

$$(\alpha f)'(c) = \alpha f'(c) \qquad (fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

$$(f \pm g)'(c) = f'(c) \pm g'(c) \qquad \left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}, \text{ assuming } g(c) \neq 0$$

Proof. These are all limit exercises, we do $\dfrac{f}{g}$.

Corollary 6.1.4. Let $f_1, f_2, \ldots, f_n : I \to \mathbf{R}$ be differentiable at $c \in I$. Then the functions $f_1 + f_2 + \cdots + f_n$ and $f_1 f_2 \cdots f_n$ are differentiable at c and:

$$(f_1 + f_2 + \cdots + f_n)'(c) = f_1'(c) + f_2'(c) + \cdots + f_n'(c)$$

$$(f_1 f_2 \cdots f_n)'(c) = f_1'(c)f_2(c) \cdots f_n(c) + f_1(c)f_2'(c) \cdots f_n(c) + \cdots + f_1(c)f_2(c) \cdots f_n'(c)$$

Example. Using the corollary, if $f(x) = x^n$, $n \in \mathbf{N}$, show that $f'(x) = nx^{n-1}$.

Example. If $f(x) = \dfrac{1}{x^n} = x^{-n}$, $n \in \mathbf{N}$, establish $f'(x) = -nx^{-n-1}$.

Notation for the derivative of $f(x)$: $f' = Df = \dfrac{df}{dx} = \dfrac{d}{dx}f(x)$

Carathéodory's Theorem 6.1.5. Let $I \subseteq \mathbf{R}$, be an interval, $f : I \to \mathbf{R}$ a function and let $c \in I$. Then f is differentiable at c if and only if there exists a function φ on I that is continuous at c and satisfies

$$f(x) - f(c) = \varphi(x)(x - c) \text{ for } x \in I.$$

If that is the case, $f'(c) = \varphi(c)$.

Proof.

Example. Determine and visualize the function φ for $f(x) = x^3$.

Note. $f'(c)$ is not the slope of the tangent line (how would you define a tangent line?); rather, the tangent line at $(c, f(c))$ is defined as the line through $(c, f(c))$ whose slope is $f'(c)$.

Theorem 6.1.6 (Chain Rule). Let $I, J \subseteq \mathbf{R}$ be intervals, and $f : I \to \mathbf{R}$, $g : J \to \mathbf{R}$ functions such that $f(I) \subseteq J$ and $c \in I$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Proof.

Example. Show the function $f : \mathbf{R} \to \mathbf{R}$ below is differentiable at 0.

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Nonexample. We can "derive" the rule for inverses with the chain rule.

Theorem 6.1.8. Let $I \subseteq \mathbf{R}$ be an interval, $f : I \to \mathbf{R}$ strictly monotone and continuous. Let $J = f(I)$ and $g : J \to \mathbf{R}$ be the strictly monotone and continuous inverse to f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $d = f(c)$ and

$$g'(d) = \frac{1}{f'(c)}$$

Proof.

Note. The assumption $f'(c) \neq 0$ is essential. Otherwise, using the chain rule, we would have $g'(d)f'(c) = 1$, which is not possible.

Theorem 6.1.9. Let $I \subseteq \mathbf{R}$ be an interval, $f : I \to \mathbf{R}$ strictly monotone. Let $J = f(I)$ and $g : J \to \mathbf{R}$ be the strictly monotone inverse to f. If f is differentiable on $\in I$ and $f'(x) \neq 0$ for $x \in I$, then g is differentiable on J and $g' = \dfrac{1}{f' \circ g}$.

Proof.

Example. If $g(x) = x^{\frac{1}{n}}$, $n \in \mathbf{N}$, show that $g'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$.

Example. If $f(x) = x^{\frac{m}{n}}$, $n \in \mathbf{N}$, $m \in \mathbf{Z}$, show that $f'(x) = \frac{m}{n}x^{\frac{m}{n}-1}$.

Example. Assuming we know $(\sin x)' = \cos x$, if $g(x) = \arcsin x$, show $g'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

6.2 The Mean Value Theorem

Definition Let I be an interval, $f : I \to \mathbf{R}$. We say that

— f *has a relative maximum at* $c \in I$ if there exists a neighborhood $V_\delta(c)$ such that $f(c) \geq f(x)$ for all $x \in V_\delta(c) \cap I$.

— f *has a relative minimum at* $c \in I$ if there exists a neighborhood $V_\delta(c)$ such that $f(c) \leq f(x)$ for all $x \in V_\delta(c) \cap I$.

We say that f *has a relative extremum at* $c \in I$ if it has a relative maximum or minimum at $c \in I$. (The terms "local" or "extreme" are often used instead of "relative" and "extremum".)

Interior Extremum Theorem 6.2.1. Let $f : I \to \mathbf{R}$ have a relative extremum at an interior point $c \in I$. If $f'(c)$ exists, then $f'(c) = 0$.

Proof.

Note. Theorem implies that if $f : I \to \mathbf{R}$ is continuous and has a relative extremum at an interior point, then $f'(c) = 0$ or $f'(c)$ does not exist.

Rolle's Theorem 6.2.3. Let $f : [a, b] \to \mathbf{R}$ be such that

a) f is continuous on $[a, b]$

b) f is differentiable on (a, b)

c) $f(a) = f(b)$.

Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Proof.

Mean Value Theorem 6.2.3. Let $f : [a, b] \to \mathbf{R}$ be such that

a) f is continuous on $[a, b]$

b) f is differentiable on (a, b).

Then there exists at least one point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Proof.

Theorem 6.2.5. Let I be an interval and $f : I \to \mathbf{R}$ a continuous function on I that is differentiable on I except possibly at endpoints. If $f'(x) = 0$ for all x where f is differentiable, then f is constant on I.

Proof.

Corollary 6.2.6. Let I be an interval and $f, g : I \to \mathbf{R}$ continuous functions on I that are differentiable on I except possibly at endpoints. If $f'(x) = g'(x)$ for all x where f and g are differentiable, then $f(x) = g(x) + C$ for all $x \in I$.

Proof.

Theorem 6.2.7. Let $f : I \to \mathbf{R}$ be differentiable on the interval I. Then:

1) f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.
2) f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.

Proof.

Note. Proof shows that if $f'(x) > 0$, then f is strictly increasing on I. The converse is not true: f can be strictly increasing on I, but $f'(x) > 0$ does not follow. Example: $f(x) = x^3$.

Example (Generalization of Bernoulli's Inequality). For $\alpha \in \mathbf{Q}$, $\alpha \geq 1$, show that $(1 + x)^\alpha \geq 1 + \alpha x$ for all $x > -1$, where the equality holds only for $x = 0$.

Read Theorem 6.2.8, First Derivative Test.

Lemma 6.2.11. Let $f : I \to \mathbf{R}$ be differentiable at $c \in I$.

1) If $f'(c) > 0$, then there exists a $\delta > 0$ such that $f(x) > f(c)$ for $x \in I \cap (c, c + \delta)$ and $f(x) < f(c)$ for $x \in I \cap (c - \delta, c)$.
2) If $f'(c) > 0$, then there exists a $\delta > 0$ such that $f(x) < f(c)$ for $x \in I \cap (c, c + \delta)$ and $f(x) > f(c)$ for $x \in I \cap (c - \delta, c)$.

Proof.

Darboux's Theorem 6.2.12 (Intermediate Value Theorem for derivatives). Let $f : [a, b] \to \mathbf{R}$ be differentiable on $[a, b]$. If k is a number strictly between $f'(a)$ and $f'(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = k$.

Proof.

Example. The function below does not satisfy the intermediate value property, so it is not the derivative of any function $f : \mathbf{R} \to \mathbf{R}$. A similar argument can be made for: if $f'(x)$ exists on an interval containing c, then f' cannot have a jump discontinuity at c. Thus, every derivative function cannot have a jump discontinuity.

$$f(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

Example. The function below is differentiable, and its derivative satisfies the intermediate value property. Note that f' is not continuous at 0, but the discontinuity is not jump.

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

6.3 L'Hospital's Rules

Theorem 6.3.1. Let $f, g : [a, b] \to \mathbf{R}$, $f(a) = g(a) = 0$, $g(x) \neq 0$ for $x \in (a, b)$. If f and g are differentiable at a and if $g'(a) \neq 0$, then the following limit exists, and

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof.

This formulation of L'Hospital's rule resolves many limits of form $\dfrac{0}{0}$.

Example. $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} =$

Example. $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{x} =$

Note. The theorem cannot be used on limits like $\displaystyle\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}}$ or $\displaystyle\lim_{x \to 0+} x \ln x = \lim_{x \to 0+} \frac{\ln x}{\frac{1}{x}}$.

Cauchy Mean Value Theorem 6.3.2. Let $f, g : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and let $g(x) \neq 0$ for $x \in (a, b)$. Then there exists a $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof.

Note. For $g(x) = x$, this is the Mean Value Theorem.

Note. Visualizing the theorem, we get a result similar to the Mean Value Theorem for parametrized curves in the plane. Let $(f(t), g(t))$, $t \in [a, b]$ be the parametrization of a curve. Then $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ can be interpreted as saying that vectors $\langle f(b) - f(a), g(b) - g(a)\rangle$ and $\langle f'(c), g'(c)\rangle$ are parallel for some $c \in (a, b)$.

L'Hospital's Rule I 6.3.3. Let $-\infty \leq a < b \leq \infty$ and let $f, g : (a, b) \to \mathbf{R}$ be differentiable on (a, b) such that $g'(x) \neq 0$ for $x \in (a, b)$. Suppose that $\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = 0$. Then

$$\text{if } \lim_{x\to a+} \frac{f'(x)}{g'(x)} = L, \text{ then } \lim_{x\to a+} \frac{f(x)}{g(x)} = L, \text{ where } L \in \mathbf{R} \cup \{-\infty, \infty\}$$

Proof.

Example. $\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}} =$

Example. $\lim_{x \to 0} \frac{x - \sin x}{x^3} =$

L'Hospital's Rule II 6.3.5. Let $-\infty \leq a < b \leq \infty$ and let $f, g : (a, b) \to \mathbf{R}$ be differentiable on (a, b) such that $g'(x) \neq 0$ for $x \in (a, b)$. Suppose that $\lim_{x \to a+} g(x) = \pm\infty$. Then

$$\text{if } \lim_{x \to a+} \frac{f'(x)}{g'(x)} = L, \text{ then } \lim_{x \to a+} \frac{f(x)}{g(x)} = L, \text{ where } L \in \mathbf{R} \cup \{-\infty, \infty\}$$

This theorem helps us deal with forms $\frac{\infty}{\infty}$. *(Proof omitted.)*

Example. $\lim_{x \to 0+} x \ln x =$

Example. $(\alpha > 0)$ $\quad \lim_{x\to\infty} \frac{x^\alpha}{e^x} =$

Example. $(\alpha > 0)$ $\quad \lim_{x\to\infty} \frac{\ln x}{x^\alpha} =$

Other indeterminate forms are $\infty - \infty$, $0 \cdot \infty$, 1^∞, 0^0, ∞^0. We can typically apply L'Hospital's rule by converting them to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ through some algebra.

Example. $\lim_{x\to 0}(1+x)^{\frac{1}{x}} =$

6.4 Taylor's Theorem

Definition. Suppose f is defined on an interval I around c and that $f'(x)$ exists for every $x \in I$. If $f'(x)$ is differentiable at c, we call its derivative the *second derivative of f at c*, denoted $f''(c)$. Similarly, if $f''(x)$ exists for every $x \in I$ and if $f''(x)$ is differentiable at c, we call the derivative of f'' at c the third derivative of f at c. We can continue in this way to get $f''(c), f'''(c), f^{(4)}(c), \dots$ Note that the existence of $f^{(n)}(c)$ requires existence of $f^{(n-1)}(x)$ on an interval around c.

Definition. Suppose $f', f'', \dots, f^{(n)}$ are all defined on an interval I containing x_0. Then we can form the *Taylor polynomial for f at x_0*:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

It is easy to see that $P_n(x)$ and $f(x)$ have the same 0th, 1st, 2nd,..., n-th derivative at x_0 so we expect that $P_n(x)$ is a good approximation of $f(x)$ for values of x close to x_0.

Taylor's Theorem 6.4.1. Let $f : [a, b] \to \mathbf{R}$ be such that $f, f', f'', \dots, f^{(n)}$ all exist and are continuous on $[a, b]$ and that $f^{(n+1)}$ exists on (a, b). If $x_0 \in [a, b]$ then for any $x \in [a, b]$ there exists a point c between x_0 and x such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Note. This looks like a higher-order version of the Mean Value Theorem, and it is: for $n = 0$ the statement is that theorem.

Definition. If we set $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$, then Taylor's theorem says that $f(x) = P_n(x) + R_n(x)$, thus we can think of $R_n(x)$ as a — presumably small — add-on to the Taylor polynomial to get the function, hence, a *remainder*. The expression $R_n(x)$ is called the *Lagrange* or *derivative form* of the remainder.

Proof.

Example. Use $P_4(x)$ to approximate $f(x) = \sqrt{x}$ near $x_0 = 4$ and estimate the size of the remainder on $[2, 6]$.

Theorem 6.4.4. Let x_0 be an interior point of an interval I, $n \geq 2$. Suppose that $f, f', f'', \ldots, f^{(n)}$ all exist and are continuous on a neighborhood of x_0 and that $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, but $f^{(n)}(x_0) \neq 0$. Then

1) If n is even and $f^{(n)}(x_0) > 0$, then there f has a relative minimum at x_0.
2) If n is even and $f^{(n)}(x_0) < 0$, then there f has a relative maximum at x_0.
3) If n is odd, then f does not have a relative extremum at x_0.

Proof.

Definition 6.4.5. Let I be an interval. A function $f : I \to \mathbf{R}$ is called *convex on* I if for any two points $x_1, x_2 \in I$ and every $t \in [0, 1]$ we have

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2).$$

Note. A convex function was called "concave up" in calculus, defined as where the graph is locally above the tangent line.

Theorem 6.4.6. Let I be an open interval and let $f : I \to \mathbf{R}$ have a second derivative on I. Then f is convex on I if and only if $f''(x) > 0$ for all $x \in I$.

Proof.

Newton's Method is a way of solving the equation $f(x) = 0$. Essentially, one "rides the tangent line" to the x-intercept. We form the recursive sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which, hopefully, converges to the zero of f.

Theorem 6.4.7. Let $I = [a, b]$ and let $f : I \to \mathbf{R}$ be twice differentiable on I. Suppose $f(a)f(b) < 0$ and there exist constants m, M such that $|f'(x)| \geq m > 0$ and $|f''(x)| \leq M$ for all $x \in I$. Let $K = \frac{M}{2m}$. Then there exists a subinterval $I^* \subseteq I$ containing the zero r of f such that for any $x_1 \in I^*$ the sequence recursively defined by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ stays in I^* and converges to r. Furthermore $|x_{n+1} - r| \leq K|x_n - r|^2$.

Note. The inequality assures rapid convergence. If $|x_n - r| < 10^{-k}$, then $|x_{n+1} - r| < K \cdot 10^{-2k}$, so method "doubles the number of correct decimals" at every step.

Proof.

Example. Use Newton's method to approximate $\sqrt[3]{2}$, the solution of $x^3 - 2 = 0$.