5.1 Continuous Functions

Definition 5.1.1. Let $A \subseteq \mathbf{R}$, $c \in A$, and let $f : A \to \mathbf{R}$ be a function. We say that f is *continuous* at c if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - c| < \delta$ and $x \in A$, then $|f(x) - f(c)| < \varepsilon$. If f is not continuous at c, we say f is *discontinuous* at c.

Note. f is continuous at c if and only if for every ε-neighborhood $V_\varepsilon(f(c))$ of $f(c)$ there is a δ-neighborhood $V_\delta(c)$ such that $f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$.

Note.

1) If c is not a cluster point of A, the definition is always satisfied.
2) If c is a cluster point of A, the definition is equivalent to $\lim_{x \to c} f(x) = f(c)$.

Theorems 5.1.3, 5.1.4: Continuity using sequences. Let $f : A \to \mathbf{R}$, $c \in A$.

1) f is continuous at c if and only if for every sequence (x_n) in A that converges to c, $\lim f(x_n) = f(c)$.
2) f is discontinuous at c if and only if there exists a sequence (x_n) in A that converges to c, but $\lim f(x_n) \neq f(c)$.

Definition 5.1.5. Let $A \subseteq \mathbf{R}$, and $B \subseteq A$. We say f is continuous on B if f is continuous at every point of B.

Example. The constant function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = b$, is continuous on $\mathbf{R}$.

Example. The identity function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x$, is continuous on $\mathbf{R}$.

Example. (Dirichlet's function) The function $f : \mathbf{R} \to \mathbf{R}$ below is discontinuous at every point.

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q} \\ 0, & \text{if } x \notin \mathbf{Q} \end{cases}$$

Example. The function $f : \mathbf{R} \to \mathbf{R}$ below is discontinuous at every point except 0.

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbf{Q} \\ -x, & \text{if } x \notin \mathbf{Q} \end{cases}$$

Example. (Thomae's function) The function $f : \mathbf{R} \to \mathbf{R}$ below is discontinuous at every rational number and continuous at every irrational number.

$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbf{Q} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ in reduced form.} \end{cases}$$

Example. The function $f : [0, \infty) \to \mathbf{R}$, $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Example. The function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = |x|$ is continuous on $\mathbf{R}$.

5.2 Combinations of Continuous Functions

Theorem 5.2.1. Let $f, g : A \to \mathbf{R}$, $c \in A$, $b \in \mathbf{R}$ and suppose f and g are continuous at c.

a) Then $f + g$, $f - g$, fg and bf are continuous at c.

b) If $g(x) \neq 0$ for all $x \in A \cap V_\delta(c)$ for some $\delta > 0$, then $\dfrac{f}{g}$ is continuous at c.

Proof.

Theorem 5.2.2. Let $f, g : A \to \mathbf{R}$, $b \in \mathbf{R}$ and suppose f and g are continuous on A.

a) Then $f + g$, $f - g$, fg and bf are continuous on A.

b) If $g(x) \neq 0$ for all $x \in A$, then $\dfrac{f}{g}$ is continuous on A.

Example. Polynomials and rational functions are continuous wherever they are defined because they are built from the constant and the identity functions using the usual algebraic operations.

Example. The function $\sin x$, $\cos x$ are continuous at every $c \in \mathbf{R}$.

Theorem 5.2.6. Let $A, B \subseteq \mathbf{R}$ and let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at point $b = f(c) \in B$, then $g \circ f$ is continuous at c.

Proof.

Theorem 5.2.7. Let $A, B \subseteq \mathbf{R}$ and let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ be functions such that $f(A) \subseteq B$. If f is continuous on A and g is continuous on B, then $g \circ f : A \to \mathbf{R}$ is continuous on A.

Example. Any single formula built using identity, constant, sine, cosine, absolute value and square root functions along with usual algebraic operations is a continuous function wherever it is defined.

5.3 Continuous Functions on Intervals

Definition 5.3.1. A function $f : A \to \mathbf{R}$ is said to be *bounded on* A if there exists a constant $M \in \mathbf{R}$ such that $|f(x)| \leq M$ for all $x \in A$.

Note. f is bounded on A if and only if $f(A)$ is a bounded set.

Boundedness Theorem 5.3.2. Let $I = [a, b]$ and let $f : I \to \mathbf{R}$ be a continuous function. Then f is bounded on I.

Proof.

Note. Both hypotheses are needed to guarantee boundedness. Give examples of a function that is not bounded on its domain for these assumptions:

a) $f : [0, \infty) \to \mathbf{R}$, f continuous.
b) $f : (0, 1] \to \mathbf{R}$, f continuous.
c) $f : [0, 1] \to \mathbf{R}$, f discontinuous at one point.

Definition 5.3.3. Let $f : A \to \mathbf{R}$. We say that

a) *f has an absolute maximum (at x^*)* if there exists an $x^* \in A$ such that $f(x^*) \geq f(x)$ for all $x \in A$.

b) *f has an absolute minimum (at x_*)* if there exists an $x_* \in A$ such that $f(x_*) \leq f(x)$ for all $x \in A$.

We say that x^* or x_* *is an absolute maximum or minimum point for f on A*, if they exist.

Maximum-Minimum Theorem 5.3.4. Let $I = [a, b]$ and let $f : I \to \mathbf{R}$ be a continuous function. Then f has an absolute maximum and absolute minimum on I.

Proof.

Location of Roots Theorem 5.3.5. Let $I = [a, b]$ and let $f : I \to \mathbf{R}$ be a continuous function. If $f(a)$ and $f(b)$ have opposite signs, then there exists a $c \in (a, b)$ such that $f(c) = 0$.

Proof.

Example. Show the equation $x - 2\cos x = 0$ has a solution in interval $[1, 2]$ and approximate the solution with accuracy 10^{-2}.

Bolzano's Intermediate Value Theorem 5.3.7. Let I be an interval and let $f : I \to \mathbf{R}$ be a continuous function. If $a, b \in I$ and $k \in \mathbf{R}$ is strictly between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = k$.

Proof.

Theorem 5.3.9. Let $I = [a, b]$ and let $f : I \to \mathbf{R}$ be a continuous function. Then $f(I)$ is the closed interval $[\inf f(I), \sup f(I)] = [\min f(I), \max f(I)]$.

Proof.

Note. The theorem does **not** say that $f([a, b])$ is $[f(a), f(b)]$ or $[f(b), f(a)]$.

5.4 Continuous Functions on Intervals

Recall: $f : A \to \mathbf{R}$ is continuous on A if, for every $c \in A$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - c| < \delta$ and $x \in A$, then $|f(x) - f(c)| < \varepsilon$. Usually, δ depends on both ε and c.

Example. For the function $f(x) = mx + b$, given ε and c, find the δ that satisfies the definition of continuity at c.

Example. For the function $f(x) = x^2$, given ε and c, find the δ that satisfies the definition of continuity at c.

Given an ε, it would be nice to have a single δ that works for every c. This is the idea behind the following definition.

Definition 5.4.1. We say a function $f : A \to \mathbf{R}$ is *uniformly continuous on* A if, for every $\varepsilon > 0$, there is a δ such that for every $x, u \in A$, if $|x - u| < \delta$, then $|f(x) - f(u)| < \varepsilon$.

Note. A function that is uniformly continuous on A is continuous on A.

Proposition 5.4.2. Let $f : A \to \mathbf{R}$. The following are equivalent:

1) f is not uniformly continuous on A.
2) There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ and u_δ satisfying $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.
3) There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$.

Proof. Left as exercise.

Uniform Continuity Theorem 5.4.3. Let $I = [a, b]$ and let $f : I \to \mathbf{R}$ be a continuous function. If f is continuous on I, then f is uniformly continuous on A.

Proof.

Definition 5.4.4. A function $f : A \to \mathbf{R}$ is said to be a *Lipschitz function* if there exists a constant $K > 0$ such that for all $x, u \in A$, $|f(x) - f(u)| \leq K|x - u|$.

Note. Every Lipschitz function is uniformly continuous on A. (Given ε, take $\delta = \frac{\varepsilon}{K}$.)

Note. f is Lipschitz if and only if $\left|\dfrac{f(x) - f(u)}{x - u}\right| \leq K$ for all $x, u \in A$, in other words, |slopes| of all secant lines are bounded by K.

Example. The function $f(x) = x^2$ is Lipschitz on a closed interval $[a, b]$, but not Lipschitz on $[0, \infty)$.

Example. The function $f(x) = \sin x$ is Lipschitz on $\mathbf{R}$.

When is a function uniformly continuous on (a, b)?

Theorem 5.4.7. If $f : A \to \mathbf{R}$ is uniformly continuous on A and (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbf{R}$.

Proof.

Continuous Extension Theorem 5.4.8. A function f is uniformly continuous on (a, b) if and only if it can be defined at a and b so that the extended function is continuous on $[a, b]$.

Proof.

Definition 5.4.9. A function $s : [a, b] \to \mathbf{R}$ is called a *step function* if there exists a collection of disjoint intervals $I_1, \dots I_n$ (open, closed or half-open) such that $\bigcup_{k=1}^{n} I_k = [a, b]$ and s is constant on I_k, $k = 1, \dots n$.

Definition. Let $f, g : A \to \mathbf{R}$. We say g *uniformly approximates f on A with accuracy* ε if $|f(x) - g(x)| < \varepsilon$ for all $x \in A$.

Theorems 5.4.10, 5.4.13, 5.4.14. Let $I = [a, b]$ and let $f : I \to \mathbf{R}$ be continuous. Then f can be uniformly approximated to any accuracy ε using a function g that is

a) a step function.
b) a continuous piecewise-linear function.
c) a polynomial.

Proof of a) and b).

5.6 Monotone and Inverse Functions

Definition. Let $f : A \to \mathbf{R}$. We say that f is

— *increasing*, if for all $x_1, x_2 \in A$, whenever $x_1 < x_2$, then $f(x_1) \leq f(x_2)$.
— *strictly increasing*, if for all $x_1, x_2 \in A$, whenever $x_1 < x_2$, then $f(x_1) > f(x_2)$.
— *decreasing*, if for all $x_1, x_2 \in A$, whenever $x_1 < x_2$, then $f(x_1) \geq f(x_2)$.
— *strictly increasing*, if for all $x_1, x_2 \in A$, whenever $x_1 < x_2$, then $f(x_1) > f(x_2)$.

A function is called *(strictly) monotone* if it is (strictly) increasing or decreasing.

The domain in this section is an interval I, finite, infinite, open, closed or half-open. We will mostly consider increasing functions — corresponding claims are valid for decreasing functions.

Note. If f is increasing, $\sup\{f(x) \mid x \in I,\ x < c\} \leq f(c) \leq \inf\{f(x) \mid x \in I,\ x > c\}$, and it is easy to find examples where the inequality is strict.

Theorem 5.6.1. Let $f : I \to \mathbf{R}$ be an increasing function, and suppose c is not an endpoint of I. Then

$$\lim_{x \to c-} f(x) = \sup\{f(x) \mid x \in I,\ x < c\} \qquad \lim_{x \to c+} f(x) = \inf\{f(x) \mid x \in I,\ x > c\}$$

Proof.

Continuous Inverse Theorem 5.6.5. Let $f : I \to \mathbf{R}$ be strictly monotone and continuous on I, and let $J = f(I)$. Then f has an inverse $f : J \to \mathbf{R}$ which is strictly monotone and continuous.

Proof. Assume f is increasing.

Example. The Continuous Inverse Theorem proves the existence of the n-th root function.

Definition 5.6.6. Knowing the n-th root exists, define $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m$, $x^{-\frac{m}{n}} = (x^{\frac{1}{n}})^{-m}$. Thus, x^r for $r \in \mathbf{Q}$ is defined for an $x > 0$.

Theorem 5.6.7. For any $x \in \mathbf{R}$, $x > 0$, $m \in \mathbf{Z}$ and $n \in \mathbf{N}$, we have $x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}}$.

Proof.

Definition For an increasing function $f : I \to \mathbf{R}$, the jump of f at c is defined as:

$$j_f(c) = \begin{cases} \lim_{x\to c+} f(x) - \lim_{x\to c-} f(x), & \text{if } c \text{ is not an endpoint of } I \\ \lim_{x\to c+} f(x) - f(c), & \text{if } c \text{ is the left endpoint of } I \\ f(c) - \lim_{x\to c-} f(x), & \text{if } c \text{ is the right endpoint of } I \end{cases}$$

Note. f is continuous at c if and only if $j_f(c) = 0$.

Theorem 5.6.4. Let $f : I \to \mathbf{R}$ be monotone. The set of points D where f is discontinuous is countable.

Proof.

Example. The function defined below is increasing and is discontinuous at every rational number. Let $q : \mathbf{N} \to \mathbf{Q}$ be a bijection (exists due to countability of $\mathbf{Q}$).

$$f(x) = \sum_{k \in \mathbf{N},\ q(k) \leq x} \frac{1}{2^k}$$