Calculus 1 — Final Exam
MAT 250, Spring 2024 — D. Ivanšić

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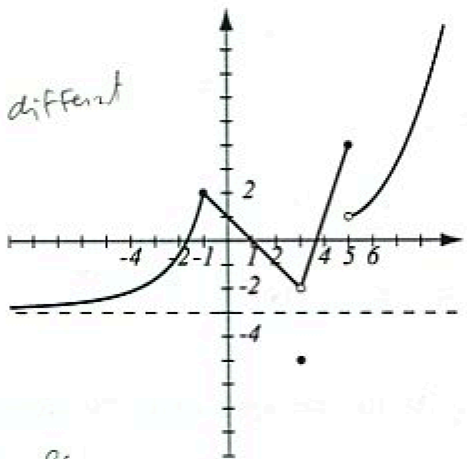
Show all your work!

1. (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.



$\lim_{x\to 5+} f(x) =$ 1

$\lim_{x\to 5-} f(x) =$ 4

$\lim_{x\to 5} f(x) =$ DNE since one-sided limits are different

$\lim_{x\to 3} f(x) =$ -2

$\lim_{x\to -\infty} f(x) =$ -3

$\lim_{x\to \infty} f(x) =$ ∞

List points where f is not continuous and explain why.

At $x=3$ $\qquad \lim_{x\to 3} f(x) = -2 \neq -5 = f(3)$

At $x=5$ $\qquad \lim_{x\to 5} f(x)$ does not exist

Find the following limits algebraically. Do not use L'Hospital's rule.

2. (6pts) $\displaystyle \lim_{x\to 3-} \frac{x+1}{2x-6} = \frac{4}{0^-} = -\infty$

$x<3$ $\qquad\qquad$ $x = 2.99$

$-\ 3\ +$ $\quad$ OR $\quad$ $2x<6$ $\quad$ OR $\quad$ $2\cdot 2.99 - 6 = 5.98 - 6 = -0.02$

$2x-6<0$

3. (6pts) $\displaystyle \lim_{x\to\infty} \frac{x^2-15}{4x^2+3x+7} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1-\overset{\to 0}{\frac{15}{x^2}}\right)}{\cancel{x^2}\left(4+\underset{\to 0}{\frac{3}{x}}+\underset{\to 0}{\frac{7}{x^2}}\right)} = \frac{1-0}{4+0+0} = \frac{1}{4}$

4. (8pts) Find $\lim_{x\to 0+} \sqrt{x}\left(2\sin\frac{1}{x} - 3\right)$. Use the theorem that rhymes with what a forest consists of.

$-1 \le \sin\frac{1}{x} \le 1$

$-2 \le 2\sin\frac{1}{x} \le 2$

$-5 \le 2\sin\frac{1}{x} - 3 \le -1$

$-5\sqrt{x} \le \sqrt{x}\left(2\sin\frac{1}{x} - 3\right) \le -\sqrt{x}$

$\lim_{x\to 0+} -\sqrt{x} = -0 = 0$

$\lim_{x\to 0+} -5\sqrt{x} = -5\cdot 0 = 0$

Equal,

so by squeeze theorem,

$\lim_{x\to 0+} \sqrt{x}\left(2\sin\frac{1}{x} - 3\right) = 0$

5. (10pts) Write the equation of the tangent line to the curve $y = x^2 e^x$ at point $(-1, \frac{1}{e})$.

$y' = 2xe^x + x^2e^x = e^x(x^2+2x)$

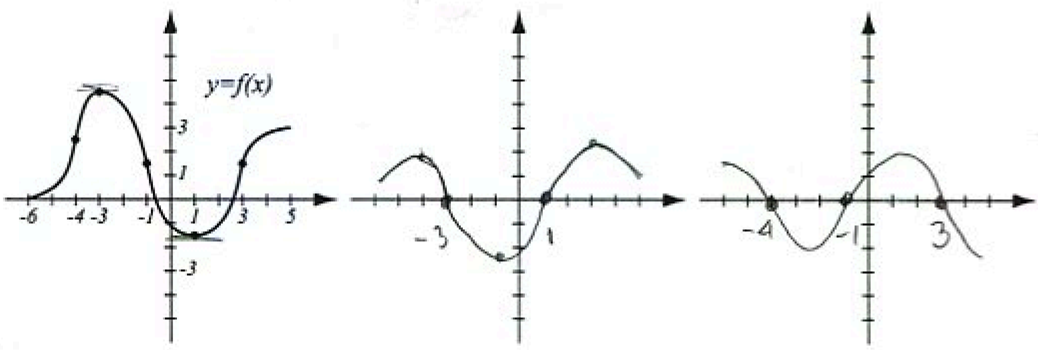
$y'(-1) = e^{-1}\left((-1)^2 + 2(-1)\right) = -\frac{1}{e}$

$y - \frac{1}{e} = -\frac{1}{e}\left(x - (-1)\right)$

$y = -\frac{1}{e}x - \frac{1}{e} + \frac{1}{e}$

$y = -\frac{1}{e}x$

6. (12pts) The graph of f is given. Use it to draw the graphs of f' and f'' in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of f. The relevant special points have been highlighted.



7. (26pts) Let $f(x) = \dfrac{x}{x^2+1}$. The domain of this function is all real numbers (you do not have to verify this). Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

d) Use information from a)–c) to sketch the graph.

$$f'(x) = \frac{1\cdot(x^2+1) - x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{-2x\,(x^2+1)^2 - (1-x^2)\,2(x^2+1)\cdot 2x}{(x^2+1)^4}$$

$$= \frac{-2x\cancel{(x^2+1)}\left(x^2+1+2(1-x^2)\right)}{(x^2+1)^{\cancel{4}\,3}} = \frac{-2x(3-x^2)}{(x^2+1)^3}$$

$$= \frac{2x(x^2-3)}{(x^2+1)^3}$$

a) $f'(x)=0$ or $f'(x)$ DNE never

$1-x^2=0$

$x^2=1 \quad x=\pm 1$

$(x^2+1)^2 \geq 0$ so sign of f' only depends on $1-x^2$

		-1		1	
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	loc. min	$\nearrow$	loc. max	$\searrow$

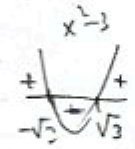
b) $f''(x)=0$ or $f''(x)$ DNE never

$2x(x^2-3)=0$

$x=0$ or $x^2-3=0$

$x=\pm\sqrt{3}$

$(x^2+1)^3 \geq 0$ so sign only depends on $x(x^2-3)$



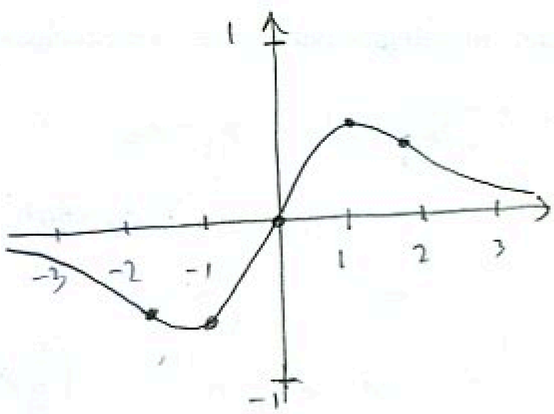
		$-\sqrt{3}$		0		$\sqrt{3}$	
x	$-$		$-$	0	$+$		$+$
x^2-3	$+$	0	$-$		$-$	0	$+$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	CD	IP	CU	IP	CD	IP	CU

c) $$\lim_{x\to\infty \,(-\infty)} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{x}{x^2\left(1+\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{1}{x\left(1+\frac{1}{x}\right)}$$

$$= \frac{1}{\infty(1+0)}_{\;-\infty} = \frac{1}{\infty}_{\;-\infty} = 0$$

d)

x	$\frac{x}{x^2+1}$
$-\sqrt{3}$	$-\frac{\sqrt{3}}{4}$
-1	$-\frac{1}{2}$
0	0
1	$\frac{1}{2}$
$\sqrt{3}$	$\frac{\sqrt{3}}{4}$



8. (12pts) Let $f(\theta) = \cos^2\theta - \sin\theta$. Find the absolute minimum and maximum values of f on the interval $[0, \frac{3\pi}{2}]$.

$f'(\theta) = 2\cos\theta(-\sin\theta) - \cos\theta$

$= -\cos\theta\,(2\sin\theta + 1)$

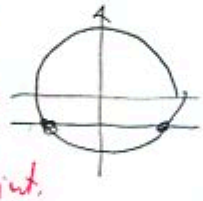
$f'(\theta) = 0$ or $f'(\theta)$ DNE
never

$\cos\theta = 0$ $\quad 2\sin\theta + 1 = 0$

$\sin\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad \theta = \frac{7\pi}{6}, \cancel{\frac{11\pi}{6}}$

not in int.



θ	$\cos^2\theta - \sin\theta$	
0	$1 - 0 = 1$	
$\frac{3\pi}{2}$	$0 - (-1) = 1$	
$\frac{\pi}{2}$	$0 - 1 = -1$	abs min
$\frac{7\pi}{6}$	$\left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$	abs max.

9. (6pts) Find $f(x)$ if $f'(x) = 3\sec^2 x + \cos x$ and $f(\frac{\pi}{4}) = -2$.

$f(x) = 3\tan x + \sin x + C$

$-2 = f(\frac{\pi}{4}) = 3\tan\frac{\pi}{4} + \sin\frac{\pi}{4} + C$

$-2 = 3\cdot 1 + \frac{\sqrt{2}}{2} + C$

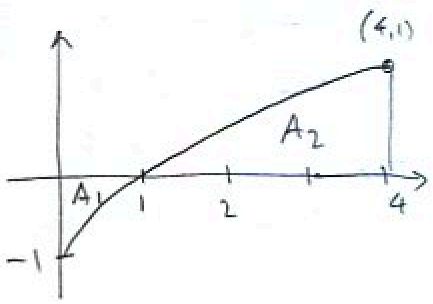
$C = -5 - \frac{\sqrt{2}}{2}$

$f(x) = 3\tan x + \sin x - 5 - \frac{\sqrt{2}}{2}$

10. (10pts) Consider the integral $\int_0^4 \sqrt{x} - 1\,dx$.

a) Use a picture and the "area" interpretation of the integral to determine whether this integral is positive or negative.

b) Use the Evaluation Theorem to find the integral and verify your conclusion from a).



a) $\int_0^4 \sqrt{x} - 1\,dx = -A_1 + A_2 = A_2 - A_1$

On the picture it appears that $A_2 > A_1$

so $A_2 - A_1 > 0$

b) $\int_0^4 \sqrt{x} - 1\,dx = \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - x\right)\Big|_0^4 = \left(\frac{2}{3}x^{\frac{3}{2}} - x\right)\Big|_0^4$

$= \frac{2}{3}\left(4^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) - (4 - 0) = \frac{2}{3}\cdot 8 - 4 = \frac{16 - 12}{3} = \frac{4}{3} > 0$

11. (10pts) The rate at which temperature in an oven is changing is $\frac{4}{\sqrt[3]{t}} + 2$ degrees Fahrenheit per minute.
a) Use the Net Change Theorem to find how much temperature changed from $t = 1$ to $t = 8$ minutes.
b) If at time $t = 1$ minutes the temperature in the oven was 170°F, what is the temperature at $t = 8$ minutes?

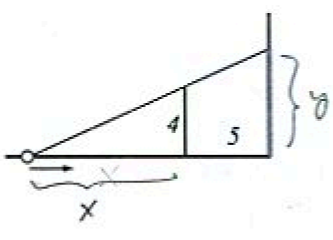
$T(t)$ = temperature at time t; $T'(t) = \frac{4}{\sqrt[3]{t}} + 2$

a) $T(8) - T(0) = \int_1^8 T'(t)\,dt = \int_1^8 4t^{-\frac{1}{3}} + 2\,dt = \left(4\frac{t^{\frac{2}{3}}}{\frac{2}{3}} + 2t\right)\Big|_1^8 = \left(6t^{\frac{2}{3}} + 2t\right)\Big|_1^8$

$= 6\left(8^{\frac{2}{3}} - 1^{\frac{2}{3}}\right) + 2(8-1) = 6(4-1) + 14 = 32\ ^\circ F$

b) $T(8) = T(1) + (T(8) - T(1)) = 170 + 32 = 202\ ^\circ F$

12. (12pts) A light source is approaching a 4-meter pole that stands 5 meters in front of a tall wall. If the light source is moving at rate 1 meters per second when it is 2 meters from the pole, how fast is the shadow of the pole on the wall growing at that moment? *Hint: similar triangles.*



Know $x' = -1$ Need: y' when $x = 2$

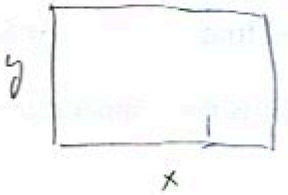
Similar triangles:

$\frac{4}{x} = \frac{y}{x+5}$ $\quad y = \frac{4(x+5)}{x} \quad \Big| \frac{d}{dt}$

$y' = 4\,\frac{x' \cdot x - (x+5)x'}{x^2} = 4 \cdot \frac{-5x'}{x^2} = \frac{-20x'}{x^2}$

When $x = 2$, $y' = \frac{-20 \cdot (-1)}{2^2} = \frac{20}{4} = 5\ m/s$

13. (16pts) Among all rectangles with area 20, find the one with the smallest perimeter.



$xy = 20$

$y = \frac{20}{x}$

$P = 2x + 2y = 2x + 2 \cdot \frac{20}{x} = 2x + \frac{40}{x}$

Job: minimize $P(x) = 2x + \frac{40}{x}$ on $(0, \infty)$

$P'(x) = 2 - \frac{40}{x^2}$

$P'(x) = 0$ or $P'(x)$ DNE (never on domain)

$2 - \frac{40}{x^2} = 0$

$2 = \frac{40}{x^2}$

$2x^2 = 40$

$x^2 = 20$

$x = \pm\sqrt{20}$

$x = \sqrt{20}$ since $x > 0$

$P''(x) = \left(2 - 40x^{-2}\right)'$

$= 80x^{-3} = \frac{80}{x^3}$

$P''(\sqrt{20}) = \frac{80}{\sqrt{20}^3} > 0$

so local min at $x = \sqrt{20}$

It is an absolute min since there is only one critical pt.

Bonus. (10pts) Draw the graph of a function that is defined for all real numbers that satisfies:

$f(-1) = -3$, $f(2) = 1$
$f'(x) > 0$ for all x in $(-1, 2)$
$f'(x) < 0$ for all x in $(-\infty, -1)$ and $(2, \infty)$
$f'(-1) = 0$, $f'(2)$ does not exist
$f''(x) > 0$ for all x in $(-\infty, 2)$ and $(2, 3)$
$f''(x) < 0$ for all x in $(3, \infty)$

		-1		2	
f'	$-$	0	$+$	DNE	$-$
f	$\searrow$	loc. min	$\nearrow$	loc max	$\searrow$

		2		3	
f''	$+$		$+$		$-$
f	Cu		Cu		CD