Calculus 1 — Exam 5
MAT 250, Spring 2024 — D. Ivanšić

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Show all your work!

Find the following antiderivatives or definite integrals.

1. (3pts) $\displaystyle\int \frac{1}{\sqrt[4]{x}}\,dx = \int x^{-\frac{1}{4}}\,dx = \frac{x^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} = \frac{x^{\frac{3}{4}}}{\frac{3}{4}} = \frac{4}{3}x^{\frac{3}{4}} + C$

2. (3pts) $\displaystyle\int \sin(2x-\pi)\,dx = -\frac{\cos(2x-\pi)}{2} + C$

3. (6pts) $\displaystyle\int (u^2 - 3\sqrt{u})u^3\,du = \int u^5 - 3u^{\frac{7}{2}}\,du = \frac{u^6}{6} - 3\frac{u^{\frac{7}{2}+1}}{\frac{7}{2}+1} = \frac{u^6}{6} - 3\cdot\frac{2}{9}u^{\frac{9}{2}}$

$\displaystyle = \frac{u^6}{6} - \frac{2}{3}u^{\frac{9}{2}} + C$

4. (5pts) $\displaystyle\int_0^{\frac{\pi}{4}} 3\sec^2\theta\,d\theta = 3\tan x\Big|_0^{\pi/4} = 3\left(\tan\frac{\pi}{4} - \tan 0\right) = 3(1-0) = 3$

5. (6pts) $\displaystyle\int_{\sqrt{e}}^{e} x - \frac{1}{x}\,dx = \left(\frac{x^2}{2} - \ln x\right)\Big|_{\sqrt{e}}^{e} = \frac{1}{2}\left(e^2 - \sqrt{e}^2\right) - \left(\ln e^2 - \ln\sqrt{e}\right)$

$\displaystyle = \frac{1}{2}(e^2 - e) - \left(2 - \frac{1}{2}\right) = \frac{e^2 - e - 3}{2}$

6. (6pts) Find $f(x)$ if $f'(x) = e^x - \cos x$ and $f(0) = 4$.

$f'(x) = e^x - \cos x$

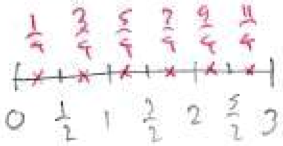
$f(x) = e^x - \sin x + C$ $\qquad\qquad f(x) = e^x - \sin x + 3$

$4 = f(0) = e^0 - \sin 0 + C$

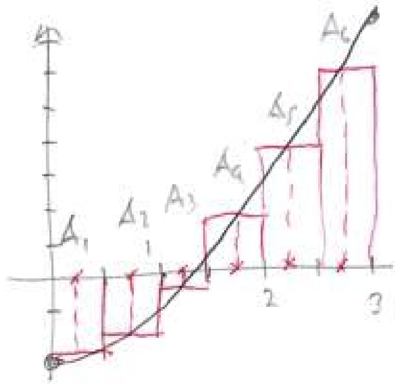
$4 = 1 + C$

$C = 3$

7. (15pts) The function $f(x) = x^2 - 2$ is given on the interval $[0, 3]$.
a) Write the Riemann sum M_6 for this function with six subintervals, taking sample points to be midpoints. Do not evaluate the expression.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does M_6 represent?



$$M_6 = \underbrace{\frac{1}{2}}_{\Delta x} \underbrace{\left(\left(\frac{1}{4}\right)^2 - 2 + \left(\frac{3}{4}\right)^2 - 2 + \left(\frac{5}{4}\right)^2 - 2 + \left(\frac{7}{4}\right)^2 - 2 + \left(\frac{9}{4}\right)^2 - 2 + \left(\frac{11}{4}\right)^2 - 2\right)}_{\sum_{i=1}^{6} f(x_i^*)}$$

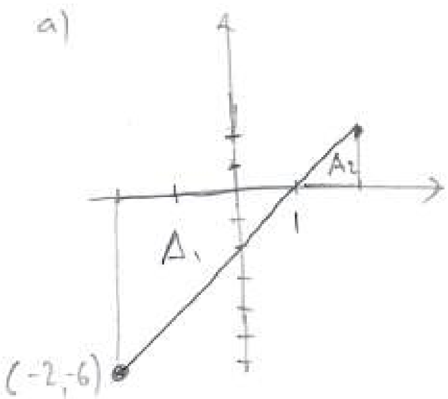


$A_1, \ldots, A_6$ = areas of rectangles

$M_6 = -A_1 - A_2 - A_3 + A_4 + A_5 + A_6$

8. (13pts) Find $\int_{-2}^{2} 2x - 2\,dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Evaluation Theorem.

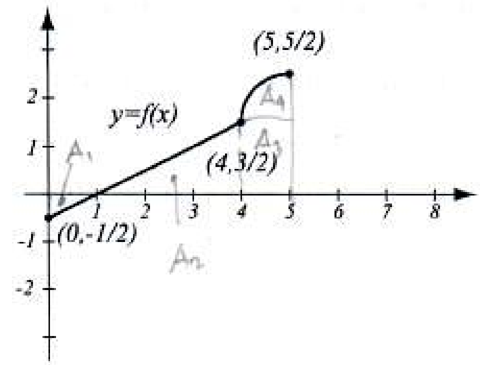
a)



$$\int_{-2}^{2} 2x - 2\,dx = -A_1 + A_2 = -\frac{1}{2}\,3 \cdot 6 + \frac{1}{2}\,1 \cdot 2$$
$$= -9 + 1 = -8$$

b) $$\int_{-2}^{2} 2x - 2\,dx = \left(2\frac{x^2}{2} - 2x\right)\Big|_{-2}^{2} = \left(x^2 - 2x\right)\Big|_{-2}^{2}$$
$$= \Big(\underbrace{2^2 - (-2)^2}_{0}\Big) - 2\big(2 - (-2)\big) = -8$$

9. (10pts) The graph of a function f, consisting of lines and parts of circles, is shown. Evaluate the integrals.



$\int_0^4 f(x)\,dx = -A_1 + A_2 = -\frac{1}{2}1\cdot\frac{1}{2} + \frac{1}{2}3\cdot\frac{3}{2}$
$= -\frac{1}{4} + \frac{9}{4} = \frac{8}{4} = 2$

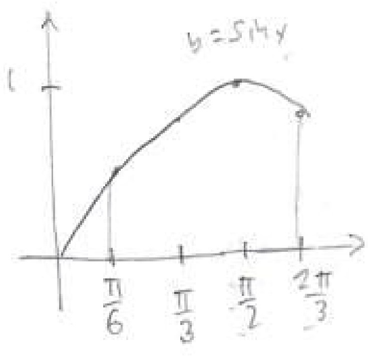
$\int_4^5 f(x)\,dx = A_3 + A_4 = 1\cdot\frac{3}{2} + \frac{1}{4}\pi\cdot 1^2 = \frac{3}{2} + \frac{\pi}{4}$

$\int_0^5 f(x)\,dx = 2 + \frac{3}{2} + \frac{\pi}{4} = \frac{14+\pi}{4}$

10. (16pts) Consider the integral $\displaystyle\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sin x\,dx$.

a) Use the inequality $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$, where $m \le f(x) \le M$ on $[a,b]$, to give an estimate of the integral. (A graph of $\sin x$ will help you find m and M.)

b) Evaluate the integral and verify your estimate from a).



a) On $\left[\frac{\pi}{6}, \frac{2\pi}{3}\right]$ we have

$\frac{1}{2} \le \sin x \le 1$

$\frac{1}{2}\left(\frac{2\pi}{3} - \frac{\pi}{6}\right) \le \int_{\pi/6}^{2\pi/3} \sin x\,dx \le 1\cdot\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)$

$\frac{\pi}{4} \le \int_{\pi/6}^{2\pi/3} \sin x\,dx \le \frac{\pi}{2}$

b) $\int_{\pi/6}^{2\pi/3} \sin x\,dx = -\cos x \Big|_{\pi/6}^{2\pi/3} = -\left(\cos\frac{2\pi}{3} - \cos\frac{\pi}{6}\right) = -\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}+1}{2}$

$1 < \sqrt{3} < 2$

$\frac{1}{2} < \frac{1+1}{2} < \frac{\sqrt{3}+1}{2} < \frac{2+1}{2} = \frac{3}{2}$

$\frac{\pi}{4} < \frac{1}{2} < \frac{\sqrt{3}+1}{2} < \frac{3}{2} < \frac{\pi}{2}$

11. (7pts) Write using sigma notation:

$$\frac{1}{4} + \frac{3}{8} + \frac{5}{16} + \cdots + \frac{13}{256} = \sum_{i=2}^{8} \frac{2i-3}{2^i}$$

2^i, $i = 2, .., 8$

$= \sum_{j=1}^{7} \frac{2j-1}{2^{j+1}}$

$2i-3$, $i = 2, .., 8$

12. (10pts) The rate at which temperature in an oven is changing is $\sqrt{t}+2$ degrees Fahrenheit per minute.
a) Use the Net Change Theorem to find how much temperature changed from $t=4$ to $t=9$ minutes.
b) If at time $t=4$ minutes the temperature in the oven was 180°F, what is the temperature at $t=9$ minutes?

a) $H(t)$ = temp. at time t

$H'(t)=\sqrt{t}+2$

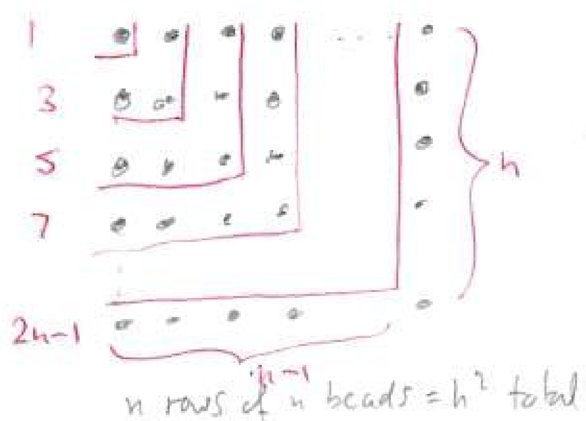
$$H(9)-H(4)=\int_4^9 H'(t)\,dt=\int_4^9 \sqrt{t}+2\,dt=\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+2t\right)\Big|_4^9=\left(\frac{2}{3}t^{\frac{3}{2}}+2t\right)\Big|_4^9$$

$$=\frac{2}{3}\left(9^{\frac{3}{2}}-4^{\frac{3}{2}}\right)+2(9-4)=\frac{2}{3}(27-8)+10=\frac{2}{3}\cdot 19+10=\frac{68}{3}\ ^\circ F$$

$\sqrt{9}^3-\sqrt{4}^3$

b) $H(9)=180+\frac{68}{3}=\frac{608}{3}\ ^\circ F=202\frac{2}{3}$

Bonus. (10pts) Show that $\sum_{i=1}^{n}(2i-1)=n^2$. (This is $1+3+5+\cdots+(2n-1)=n^2$.)
Use either a picture with beads (what is a good way to picture n^2 beads?) that is cleverly divided up, or show it algebraically, for which you may find the formula $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$ useful.



OR

$$\sum_{i=1}^{n}(2i-1)=2\sum_{i=1}^{n} i-\sum_{i=1}^{n} 1$$

$$=2\cdot\frac{n(n+1)}{2}-n=n^2+n-n=n^2$$