**Calculus 1 — Exam 4**
**MAT 250, Spring 2024 — D. Ivanšić**

**Name:** Saul Ocean

*Show all your work!*

**1.** (32pts) Let $f(x) = \dfrac{x}{x^2+4}$. The domain of this function is all real numbers (you do not have to verify this). Draw an accurate graph of $f$ by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
d) Use information from a)–c) to sketch the graph.

$$f'(x) = \frac{1\cdot(x^2+4) - x\cdot 2x}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$$

$$f''(x) = \frac{-2x(x^2+4)^2 - (4-x^2)2(x^2+4)2x}{(x^2+4)^4}$$

$$= \frac{-2x(x^2+4)\left(x^2+4+(4-x^2)2\right)}{(x^2+4)^4}$$

$$= \frac{-2x(12-x^2)}{(x^2+4)^3} = \frac{2x(x^2-12)}{(x^2+4)^3}$$

$f'(x)=0$ $\quad 4-x^2=0$
$x^2=4 \quad x=\pm 2$

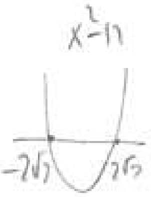
sign of $f'$ = sign of $4-x^2$

$4-x^2$

| | | $-2$ | | $2$ | |
|---|---|---|---|---|---|
| $f'(x)$ | $-$ | $0$ | $+$ | $0$ | $-$ |
| $f(x)$ | $\searrow$ | loc. min | $\nearrow$ | loc. max | $\searrow$ |

$f''(x)=0 \quad x=0$ or $x^2-12=0$
$x=\pm\sqrt{12}=\pm 2\sqrt{3}$

$x^2-12$



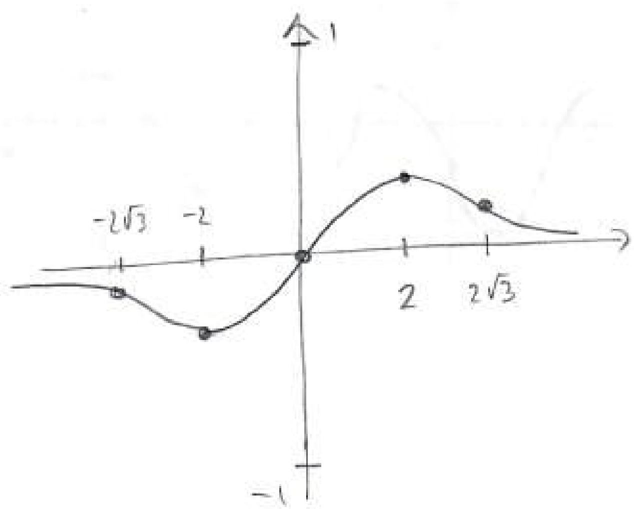
| | | $-2\sqrt3$ | | $0$ | | $2\sqrt3$ | |
|---|---|---|---|---|---|---|---|
| $x$ | $-$ | | $-$ | $0$ | $+$ | | $+$ |
| $x^2-12$ | $+$ | $0$ | $-$ | | $-$ | $0$ | $+$ |
| $f''(x)$ | $-$ | $0$ | $+$ | $0$ | $-$ | $0$ | $+$ |
| $f(x)$ | CD | IP | CU | IP | CD | IP | CU |

c) $\lim\limits_{x\to\infty} \dfrac{x}{x^2+4} = \lim\limits_{x\to\infty} \dfrac{x}{x^2(1+\frac{4}{x^2})}$

$= \lim\limits_{x\to\infty} \dfrac{1}{x(1+\frac{4}{x^2})} = \dfrac{1}{\infty\cdot 1} = 0$

Same for $\lim\limits_{x\to-\infty} f(x)$

| $x$ | $\frac{x}{x^2+4}$ |
|---|---|
| $-2$ | $\frac{-2}{8} = -\frac14$ |
| $2$ | $\frac{2}{8} = \frac14$ |
| $-2\sqrt3$ | $\frac{-2\sqrt3}{12+4} = -\frac{\sqrt3}{8}$ |
| $2\sqrt3$ | $\frac{2\sqrt3}{12+4} = \frac{\sqrt3}{8}$ |



**2.** (18pts) Let $f(x) = \sin^2\theta - \cos\theta$. Find the absolute minimum and maximum values of $f$ on the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

$f'(\theta) = 2\sin\theta\cos\theta - (-\sin\theta)$

$f'(\theta) = \sin\theta\,(2\cos\theta + 1)$

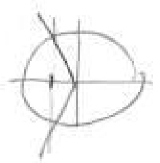
$f'(\theta) = 0$

$\sin\theta = 0$ or $2\cos\theta + 1 = 0$

$\theta = \cancel{0}, \pi, \cancel{2\pi}$ $\quad \cos\theta = -\frac{1}{2}$

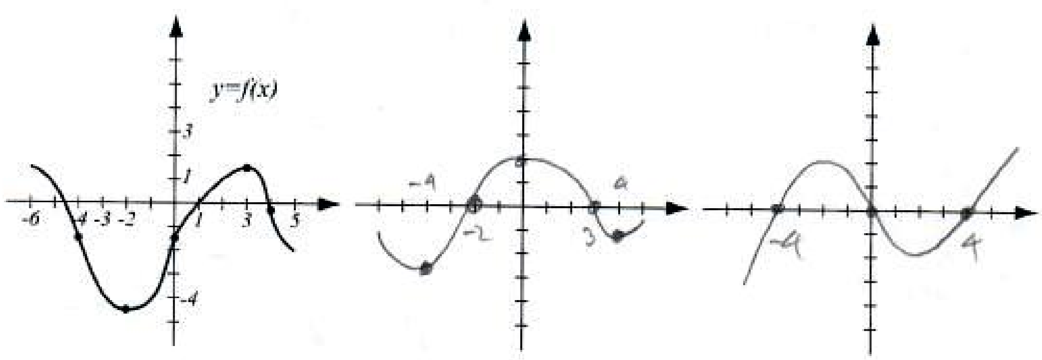
not in interval

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$



| $\theta$ | $\sin^2\theta - \cos\theta$ | |
|---|---|---|
| $\frac{\pi}{2}$ | $1 - 0 = 1$ | abs min |
| $\frac{3\pi}{2}$ | $1 - 0 = 1$ | |
| $\pi$ | $0 - (-1) = 1$ | |
| $\frac{2\pi}{3}$ | $\left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ | abs max |
| $\frac{4\pi}{3}$ | $\left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ | |

**3.** (14pts) The graph of $f$ is given. Use it to draw the graphs of $f'$ and $f''$ in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of $f$. The relevant special points have been highlighted.



**4.** (14pts) Consider $f(x) = x^3 - 4x^2$ on the interval $[0, 2]$.
a) Verify that the function satisfies the assumptions of the Mean Value Theorem.
b) Find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.

a) $f(x)$ is continuous and differentiable on $\mathbb{R}$ so in particular on $[0,2]$.

b) $$\frac{f(2)-f(0)}{2-0} = \frac{(8-16)-0}{2-0} = \frac{-8}{2} = -4$$

$$f'(x) = 3x^2 - 8x$$

$$3x^2 - 8x = -4$$

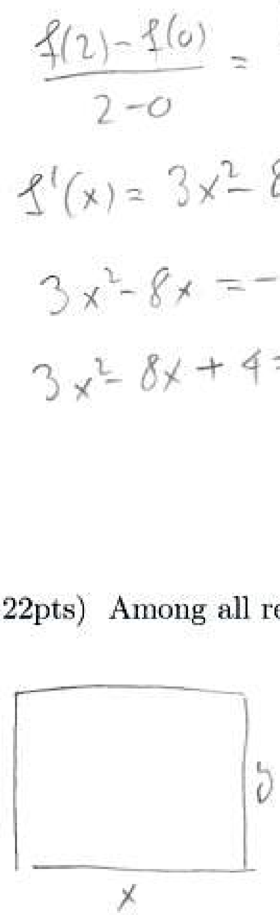
$$3x^2 - 8x + 4 = 0$$

$64 - 48 = 16$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 3 \cdot 4}}{2 \cdot 3} = \frac{8 \pm \sqrt{16}}{6} = \frac{8 \pm 4}{6} = 2, \frac{2}{3}$$

2 is not in $(0,2)$
$\frac{2}{3}$ is in $(0,2)$

**5.** (22pts) Among all rectangles with area 24, find the one with the smallest perimeter.




$$xy = 24 \Rightarrow y = \frac{24}{x}$$

$$P = 2x + 2y = 2x + \frac{48}{x}$$

Job: minimize $P(x)$ on $(0, \infty)$

$$P'(x) = 2 - \frac{48}{x^2}$$

$$2 - \frac{48}{x^2} = 0 \quad | \cdot x^2$$

$$2x^2 - 48 = 0$$

$$x^2 = 24$$

$$x = \pm\sqrt{24} = \pm 2\sqrt{6}$$

only $2\sqrt{6}$ is in $(0, \infty)$

$$P''(x) = (2 - 48x^{-2})' = 96x^{-3} = \frac{96}{x^3}$$

$P''(2\sqrt{6}) > 0$ so $P$ has a local min at $x = 2\sqrt{6}$

Since there is only one critical point, it is the absolute minimum

**Bonus.** (10pts) Draw the graph of a function that is defined for all real numbers that satisfies:

$f(-1) = -3$, $f(2) = 1$
$f'(x) > 0$ for all $x$ in $(-1, 2)$
$f'(x) < 0$ for all $x$ in $(-\infty, -1)$ and $(2, \infty)$
$f'(-1) = 0$, $f'(2)$ does not exist
$f''(x) > 0$ for all $x$ in $(-\infty, 2)$ and $(\cancel{3, \infty})$ $2,3$
$f''(x) < 0$ for all $x$ in $(\cancel{-2, 3})$ $(3, \infty)$