Calculus 1 — Exam 3
MAT 250, Spring 2024 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (4pts) $\dfrac{d}{dx}x^2 5^x = 2x\cdot 5^x + x^2\cdot \ln 5\cdot 5^x = 5^x(2x+x^2\ln 5)$

2. (6pts) $\dfrac{d}{d\theta}e^\theta \sin(2\theta) = e^\theta \sin(2\theta) \cdot e^\theta \cos(2\theta)\cdot 2$

$= e^\theta(\sin(2\theta) + 2\cos(2\theta))$

3. (7pts) $\dfrac{d}{du}\dfrac{e^u - \sin u}{e^u + \sin u} = \dfrac{(e^u - \cos u)(e^u + \sin u) - (e^u - \sin u)(e^u + \cos u)}{(e^u+\sin u)^2}$

$= \dfrac{\cancel{e^{2u}} - e^u\cos u + e^u \sin u - \cancel{\sin u\cos u} - (\cancel{e^{2u}} - e^u\sin u + e^u\cos u - \cancel{\sin u\cos u})}{(e^u+\sin u)^2} = \dfrac{2e^u(\sin u - \cos u)}{(e^u+\sin u)^2}$

4. (7pts) $\dfrac{d}{dx}\ln\dfrac{\sin^2 x}{\tan x} = \dfrac{d}{dx}\left(\ln \sin^2 x - \ln \tan x\right) = \dfrac{d}{dx}\left(2\ln\sin x - \ln\tan x\right)$

$= 2\dfrac{1}{\sin x}\cos x - \dfrac{1}{\tan x}\cdot\sec^2 x = 2\cot x - \dfrac{\cancel{\cos x}}{\sin x}\cdot\dfrac{1}{\cancel{\cos^2 x}} = 2\cot x - \dfrac{1}{\sin x\cos x}$

5. (7pts) $\dfrac{d}{dv}\arcsin\sqrt{1-v^2} = \dfrac{1}{\sqrt{1-\sqrt{1-v^2}^{\,2}}}\cdot\dfrac{1}{\cancel{2}\sqrt{1-v^2}}\cdot(-\cancel{2}v) = \dfrac{1}{\sqrt{1-(1-v^2)}}\cdot\dfrac{-v}{\sqrt{1-v^2}}$

$= \dfrac{-v}{\sqrt{v^2}\sqrt{1-v^2}} = \dfrac{-v}{|v|\sqrt{1-v^2}} = \left[\begin{matrix}\text{if } v>0\\ |v|=v\end{matrix}\right] = -\dfrac{1}{\sqrt{1-v^2}}$

6. (9pts) Use logarithmic differentiation to find the derivative of $y = (\cos x)^{\cos x}$.

$y = (\cos x)^{\cos x}$

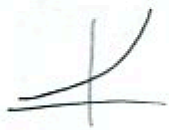
$\ln y = \ln(\cos x)^{\cos x} = \cos x\ln(\cos x) \quad \Big| \dfrac{d}{dx}$

$\dfrac{1}{y}\cdot y' = -\sin x\ln\cos x + \cos x\cdot\dfrac{1}{\cos x}(-\sin x)$

$y' = y(-\sin x\ln\cos x - 1) = -(\cos x)^{\cos x}\sin x(\ln\cos x + 1)$

Find the limits algebraically. Graphs of basic functions will help, as will L'Hospital's rule, where appropriate.

7. (2pts) $\lim_{x\to\infty} e^{-0.5x} =$ $e^{-\infty} = 0$



$\ln\infty - \ln\infty = \infty - \infty$

8. (7pts) $\lim_{x\to\infty} (\ln(x+2) - \ln(x^2-1)) =$ $\lim_{x\to\infty} \ln \frac{x+2}{x^2-1} = \ln \lim_{x\to\infty} \frac{x+2}{x^2-1}$

$= \ln \lim_{x\to\infty} \frac{x(1+\frac{2}{x})}{x^2(1-\frac{1}{x^2})} = \ln \lim_{x\to\infty} \frac{1}{x} \cdot \frac{1+\frac{2}{x}}{1-\frac{1}{x^2}} = \ln(0+) \frac{1+0}{1-0}$

since $x>0$, $\frac{1}{x}>0$ $\quad = \ln 0+ \quad = -\infty$

9. (7pts) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} =$ $\left[\frac{e^0-1-0}{0^2} = \frac{1-1}{0} = \frac{0}{0}\right] \overset{L'H}{=} \lim_{x\to 0} \frac{e^x-1}{2x} \cdot \left[\frac{e^0-1}{2\cdot 0} = \frac{1-1}{0} = \frac{0}{0}\right]$

$\overset{L'H}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$

10. (9pts) $\lim_{x\to\infty} x^2 \sin\frac{1}{x^2} =$ $\left[\begin{matrix}\infty \cdot \sin\frac{1}{\infty} = \\ \infty \cdot \sin 0 = \infty \cdot 0\end{matrix}\right] = \lim_{x\to\infty} \frac{\sin\frac{1}{x^2}}{\frac{1}{x^2}} = \left[\frac{\sin\frac{1}{\infty}}{\frac{1}{\infty}} = \frac{\sin 0}{0} = \frac{0}{0}\right]$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\cos\frac{1}{x^2}\cdot\cancel{(-2x^{-3})}}{\cancel{-2x^{-3}}} = \lim_{x\to\infty} \cos\frac{1}{x^2} = \cos\frac{1}{\infty} = \cos 0 = 1$

$\left(\frac{1}{x^2}\right)' = (x^{-2})' = -2x^{-3}$

11. (8pts) $\lim_{x\to\infty} (x^4+3)^{\frac{1}{2x}} =$ $\lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^0 = 1$

$y = (x^4+3)^{\frac{1}{2x}}$

$\ln y = \ln(x^4+3)^{\frac{1}{2x}} = \frac{1}{2x}\ln(x^4+3)$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(x^4+3)}{2x} = \left[\frac{\ln\infty}{\infty} = \frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x^4+3}\cdot 4x^3}{2} = \lim_{x\to\infty} \frac{4x^3}{x^4+3} \overset{L'H}{=} \lim_{x\to\infty} \frac{12x^2}{4x^3}$

$\to\infty$, $\to\infty$

$= \lim_{x\to\infty} \frac{3}{x} = \frac{3}{\infty} = 0$

12. (11pts) Let $f(x) = \arctan x$.

a) Write the linearization of $f(x)$ at $a = 0$.

b) Use the linearization to estimate $\arctan(-\frac{1}{2})$.

c) In the same coordinate system, draw graphs of the function and the linearization and determine if the estimate in b) is an overestimate or underestimate of $\arctan(-\frac{1}{2})$.

a) $f(0) = \arctan 0 = 0$

$f'(x) = \dfrac{1}{1+x^2}$

$f'(0) = \dfrac{1}{1+0^2} = 1$

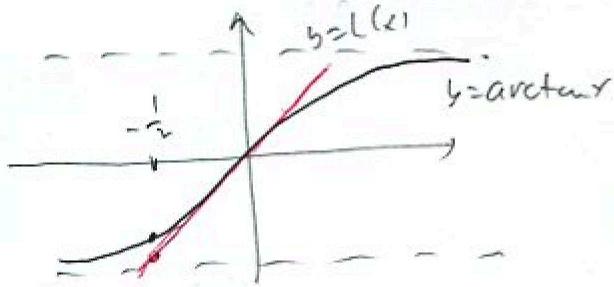
$L(x) = f(0) + f'(0)(x-0)$

$= 0 + 1 \cdot x$

$L(x) = x$

b) $L(-\frac{1}{2}) = -\dfrac{1}{2}$

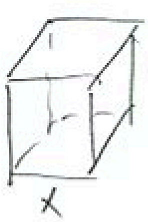
c)



$L(-\frac{1}{2}) < \arctan -\frac{1}{2}$

so it is an underestimate

13. (9pts) A cube is measured to have side length of 3 meters, with maximum error 0.5 centimeters. Use differentials to estimate the maximum possible error when computing the surface area of the cube.



S = surface area $= 6x^2$

$\Delta S \simeq dS$

$dS = S'(x)\,dx$

$= 12x\,dx$

when $x = 3$ m

$dx = 0.5$ cm $= 0.005$ m

we have $dS = 12 \cdot 3 \cdot 0.005$

$= \dfrac{36 \cdot 5}{1000} = \dfrac{36}{200} = \dfrac{9}{50}$ m^2

14. (7pts) Let $f(x) = x - \sqrt[3]{x}$. Use the theorem on derivatives of inverses to find $(f^{-1})'(6)$.

$f(x) = x - \sqrt[3]{x}$

$f'(x) = 1 - \frac{1}{3}x^{-\frac{2}{3}}$

$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(8)} = \frac{1}{1-\frac{1}{3}8^{-\frac{2}{3}}} = \frac{1}{1-\frac{1}{12}} = \boxed{\frac{12}{11}}$

$f^{-1}(6)$ is sol to $x - \sqrt[3]{x} = 6$

$x = 8$ by guessing

$8^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{8}^2} = \frac{1}{4}$

Bonus. (10pts) Find the limit.

$\lim_{x \to 0+} \ln x \ln(x+1) =$ $[-\infty \cdot \ln 1 = \infty \cdot 0$ indeterminate$]$

$= \lim_{x\to 0+} \frac{\ln(x+1)}{\frac{1}{\ln x}} \to \left[\frac{\ln 1}{\frac{1}{\ln 0+}} = \frac{0}{\frac{1}{-\infty}} = \frac{0}{0}\right] \overset{L'H}{=} \lim_{x\to 0+} \frac{\frac{1}{x+1}}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}}$

$\frac{d}{dx}(\ln x)^{-1} = -(\ln x)^{-2}\cdot\frac{1}{x}$

$= \lim_{x\to 0+} \underbrace{\frac{1}{x+1}}_{\to 1} \cdot (-x(\ln x)^2) = (-1)\cdot \lim_{x\to 0} \frac{(\ln x)^2}{\frac{1}{x}} \overset{L'H}{=} -\lim_{x\to 0} \frac{2\ln x \cdot \frac{1}{x}}{-\frac{1}{x^2}}$.

$(-\infty)^2 = \infty$

$\frac{1}{0+} = \infty$

$= \lim_{x\to 0+} 2x\ln x = 2\lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} \overset{L'H}{=} 2\lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 2\lim_{x\to 0+} -\frac{x^2}{x} = -2\lim_{x\to 0+} x = 2\cdot 0$

$\ln x \to -\infty$, $\frac{1}{x} \to \frac{1}{0+} = \infty$

$= 0$