Calculus 1 — Exam 2
MAT 250, Spring 2024 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{dx}\left(3x^6 - \frac{4}{x^4} + \frac{5}{\sqrt[6]{x}} + \pi^3\right) = \frac{d}{dx}\left(3x^6 - 4x^{-4} + 5x^{-\frac{1}{6}} + \pi^3\right)$

$= 18x^5 + 16x^{-5} - \frac{5}{6}x^{-\frac{7}{6}}$

2. (5pts) $\frac{d}{dx}(x^2+1)\cos x = 2x\cos x + (x^2+1)(-\sin x)$

$= -x^2\sin x + 2x\cos x - \sin x$

3. (6pts) $\frac{d}{du}\frac{(u+1)^2}{(u-4)^3} = \frac{2(u+1)(u-4)^3 - (u+1)^2 3(u-4)^2}{((u-4)^3)^2} = \frac{(u+1)\cancel{(u-4)^2}(2(u-4)-3(u+1))}{(u-4)^{\cancel{6}\,4}}$

$= \frac{(u+1)(2u-8-3u-3)}{(u-4)^4} = \frac{(u+1)(-u-11)}{(u-4)^4}$

4. (6pts) $\frac{d}{d\theta}\frac{\cos\theta}{\cos\theta - \sin\theta} = \frac{-\sin\theta(\cos\theta - \sin\theta) - \cos\theta(-\sin\theta - \cos\theta)}{(\cos\theta - \sin\theta)^2}$

$= \frac{-\cancel{\sin\theta\cos\theta} + \sin^2\theta + \cancel{\cos\theta\sin\theta} + \cos^2\theta}{(\cos\theta - \sin\theta)^2} = \frac{1}{(\cos\theta - \sin\theta)^2}$

$=1$

5. (6pts) $\frac{d}{dz}\tan\sqrt{\sec z} = \sec^2\sqrt{\sec z}\cdot\frac{1}{2\sqrt{\sec z}}\sec z\tan z$

$= \frac{\sec z\tan z\ \sec^2\sqrt{\sec z}}{2\sqrt{\sec z}}$

6. (7pts) Let $y(x) = x^4$.
a) Write the first four derivatives of y.
b) What is the n-th derivative of y for $n \geq 5$?

a

a) $y' = 4x^3$

$y'' = 4\cdot3x^2$

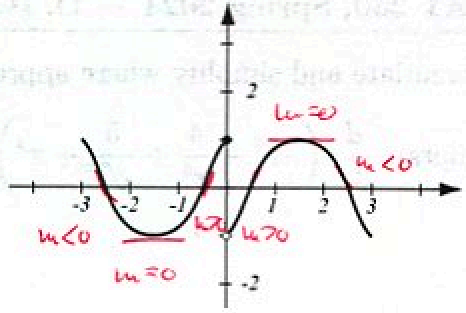
$y''' = 4\cdot3\cdot2x^2$

$y^{(4)} = 4\cdot3\cdot2\cdot1$

$y^{(5)}(x) = 0$

$y^{(n)}(x) = 0$ for $n > 5$

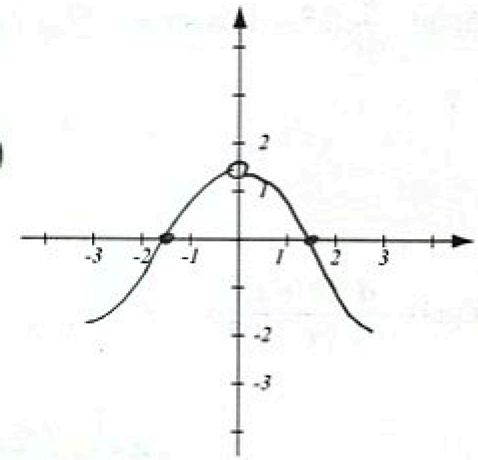
7. (10pts) The graph of the function $f(x)$ is shown at right.



a) Where is $f(x)$ not differentiable? Why?
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.

a) At $x=0$, function is not continuous, so not differentiable

b)



8. (12pts) Let $f(x) = 2x^2 - 5x + 1$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using differentiation rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(2, -1)$.

a) $f'(a) = \lim\limits_{x \to a} \dfrac{2x^2-5x+1-(2a^2-5a+1)}{x-a} = \lim\limits_{x\to a} \dfrac{2(x^2-a^2)-5(x-a)}{x-a}$

$= \lim\limits_{x\to a} \dfrac{2(x-a)(x+a)-5(x-a)}{x-a} = \lim\limits_{x\to a} \dfrac{\cancel{(x-a)}(2(x+a)-5)}{\cancel{x-a}} = 2(a+a)-5$
$= 4a-5$

$f'(x) = 4x-5$

b) $f'(x) = 2\cdot 2x - 5 = 4x-5$, agrees

c) $f'(2) = 4\cdot 2 - 5 = 3$ $y-(-1) = 3(x-2)$
$y = 3x-6-1$
$y = 3x-7$

9. (12pts) Let $g(x) = f(x^3)$ and $h(x) = \dfrac{(f(x))^2}{x}$.

a) Find the general expressions for $g'(x)$ and $h'(x)$.
b) Use the table of values at right to find $g'(1)$ and $h'(3)$.

x	1	2	3	4
$f(x)$	3	2	5	-3
$f'(x)$	2	-1	-4	2

a) $g'(x) = f'(x^3)\cdot 3x^2$

$h'(x) = \dfrac{2f(x)f'(x)\cdot x - f(x)^2\cdot 1}{x^2} = \dfrac{2xf(x)f'(x) - f(x)^2}{x^2}$

b) $g'(1) = f'(1)\cdot 3\cdot 1^2 = 2\cdot 3 = 6$ $\quad h'(3) = \dfrac{2\cdot 3\cdot f(3)f'(3) - f(3)^2}{3^2} = \dfrac{2\cdot 3\cdot 5\cdot(-4) - 5^2}{9}$

$= \dfrac{-120-25}{9} = -\dfrac{145}{9}$

10. (6pts) An ball thrown upwards has position (in feet, t in seconds) given by the formula $s(t) = -16t^2 + 40t$.
a) Write the formula for the velocity of the ball at time t.
b) What is the highest altitude that the ball reaches?

a) $v(t) = -32t + 40$

b) highest altitude is when $v(t) = 0$

$-32t + 40 = 0$ $\quad s(\frac{5}{4}) = -16\cdot\frac{25}{16} + 40\cdot\frac{5}{4} = -25 + 50 = 25$ ft

$-32t = -40$

$t = \frac{40}{32} = \frac{5}{4}$

11. (10pts) Use implicit differentiation to find y' in general

$\sin(xy) = \sin(x^2) + \sin(y^2)$ $\quad \Big| \frac{d}{dx}$

$\cos(xy)\cdot(xy)' = \cos(x^2)\cdot 2x + \cos(y^2)\cdot 2yy'$

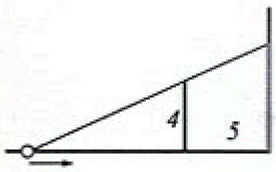
$\cos(xy)(y + xy') = 2x\cos(x^2) + 2y\cos(y^2)y'$

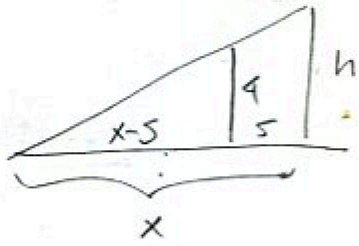
$y\cos(xy) + x\cos(xy)y' = 2x\cos(x^2) + 2y\cos(y^2)y'$

$x\cos(xy)y' - 2y\cos(y^2)y' = 2x\cos(x^2) - y\cos(xy)$

$y' = \dfrac{2x\cos(x^2) - y\cos(xy)}{x\cos(xy) - 2y\cos(y^2)}$

12. (14pts) A light source is approaching a 4-meter pole that stands 5 meters in front of a tall wall. If the light source is moving at rate 0.5 meters per second when it is 3 meters from the pole, how fast is the shadow of the pole on the wall growing at that moment? *Hint: similar triangles.*





Know $x' = -0.5$ m/s when $x = 8$

Need h' when $x = 8$

From similar triangles:

$$\frac{4}{x-5} = \frac{h}{x}$$

$$h = \frac{4x}{x-5} \quad \Big| \frac{d}{dt}$$

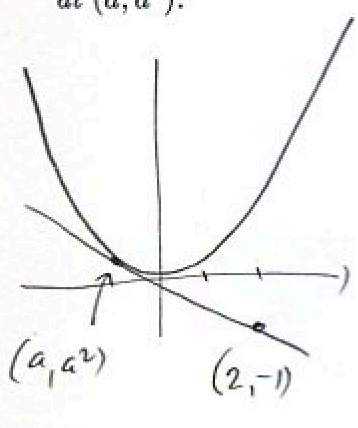
$$h' = \frac{4x'(x-5) - 4x\cancel{x'}}{(x-5)^2}$$

$$h' = \frac{-20x'}{(x-5)^2}$$

When $x = 8$

$$h' = \frac{-20 \cdot (-\frac{1}{2})}{(8-5)^2} = \frac{10}{9} \text{ m/s}$$

Bonus. (10pts) Find the points on the curve $y = x^2$ at which the tangent line passes through the point $(2, -1)$, which is not on the curve. *Hint: look for a point (a, a^2) on the curve so that the slope of the line through (a, a^2) and $(2, -1)$ is equal to the slope of the tangent line at (a, a^2).*



Slope of line through $(a,a^2), (2,-1)$ = slope of tan. line at (a,a^2)

$$\frac{a^2 - (-1)}{a-2} = 2a$$

$$a^2 + 1 = 2a(a-2)$$

$$a^2 + 1 = 2a^2 - 4a$$

$$a^2 - 4a - 1 = 0$$

$$a = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

The points are

$(2+\sqrt{5}, (2+\sqrt{5})^2)$

$(2-\sqrt{5}, (2-\sqrt{5})^2)$