Calculus 1 — Lecture notes
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3.1 Exponential Functions

Recall that the function $f(x) = a^x$, $a > 0$, $a \neq 1$ is called an exponential function.

Graph:

From the graphs we can see the most important facts about exponential functions.

Continuity:

Domain = Range =

$\lim_{x\to\infty} a^x =$ $\lim_{x\to-\infty} a^x =$

Using above facts, we can find limits involving a^x:

Example. $\lim_{x\to 3} 5^{\frac{x^2-4x+3}{x-3}} =$

Example. $\lim_{x\to 4+} e^{\frac{2}{8-2x}} =$

Example. $\lim_{x\to\infty} \dfrac{3^x - 1}{3^{2x} + 5 \cdot 3^x - 3} =$

The number e can be defined in several ways, here are two:

1) $e = \lim_{x \to 0}(1+x)^{\frac{1}{x}} \approx$

x	$(1+x)^{\frac{1}{x}}$
0.1	
0.01	
0.001	
10^{-4}	
10^{-5}	
10^{-6}	

2) Let m_a = slope of tangent line to graph of $y = a^x$ at $x = 0$. It can be numerically found that

$$m_2 < 1 \text{ and } m_3 > 1$$

Since m_a varies continuously with a, by the Intermediate Value Theorem there is a number a such that $m_a = 1$.

In this approach, we can define e as the number such that the graph of $y = e^x$ has a tangent line at $x = 0$ whose slope is 1.

3.2 Inverse Functions and Logarithms

Definition. A function is *one-to-one* if it sends different x's to different y's, that is

$$\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2)$$

In other words, it never happens that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Thus, no two points on the graph with different x-coordinates have the same y-coordinate. This is the idea of the:

Horizontal line test. A function is one-to-one if and only if no horizontal line intersects it more than once.

Example. Are the following graphs of one-to-one functions?

One-to-one functions are important because they are the only functions that have inverses: For every y in the range of f, we can define:

$$f^{-1}(y) = \text{ the } x \text{ such that } f(x) = y$$

Example. The function $f(x) = x^2$ is not one-to-one, but we have learned that its inverse is $\sqrt{x}$. What gives?

In general, functions that are not one-to-one, like $\sqrt{\ }$, sin, cos, tan,... are turned into one-to-one functions in the same way, by *restricting the domain*.

Inverse functions satisfy: $f^{-1}(f(x)) = x \qquad f(f^{-1}(x)) = x$

The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

How to find the derivative of f^{-1}:

Theorem. If f is a one-to-one differentiable function then its inverse f^{-1} is differentiable, and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Example. Use the theorem to find the derivative of $\sqrt[3]{x}$.

Example. Let $f(x) = 2x + \cos x$. Use the theorem to find $(f^{-1})'(1)$.

Definition. The exponential function $f(x) = a^x$ is one-to-one, so has an inverse called the *logarithmic function*: $f^{-1}(x) = \log_a x$.

Note. Think of $\log_a x$ as the answer to the question $a^? = x$, in other words,

$$y = \log_a x \text{ is the same as saying } a^y = x$$

Properties of logarithmic functions

$\log_a a^x = x$ $\qquad$ $a^{\log_a x} = x$

are just $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for $f(x) = a^x$ and $f^{-1}(x) = \log_a x$.

Property	Related exponential property
$\log_a(xy) = \log_a x + \log_a y$	$a^{u+v} = a^u \cdot a^v$
$\log_a \dfrac{x}{y} = \log_a x - \log_a y$	$a^{u-v} = \dfrac{a^u}{a^v}$
$\log_a x^r = r \log_a x$	$(a^u)^v = a^{uv}$

Change of base formula:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Special bases: $a = e$, we write $\log_e x = \ln x$
$a = 10$, we write $\log_{10} x = \log x$

From the graph of a^x, $a > 1$, we get the graph of $\log_a x$ and can see all the important facts about it:

Domain =

Range =

$\lim_{x\to\infty} \log_a x =$

$\lim_{x\to 0+} \log_a x =$

Example. $\lim_{x\to 0+} \log_2(\sin x) =$

3.3 Derivatives of Exponential and Logarithmic Functions

Derivative of the exponential function $f(x) = a^x$

Theorem. $\dfrac{d}{dx}\, e^x = e^x \qquad \dfrac{d}{dx}\, a^x = \ln a \cdot a^x$

Example. $\dfrac{d}{dx}\,(\sqrt{x}e^x) =$

Example. $\dfrac{d}{dx}\, e^{\cos x} =$

Example. $\dfrac{d}{dx}\,(x^2+3x)2^{4x} =$

Example. $\dfrac{d}{dx}\,\dfrac{x}{e^x} =$

Derivative of the logarithmic function $\log_a x$

Set $f(x) = a^x$, so $f^{-1}(x) = \log_a x$.

Theorem. $$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

Example. $\frac{d}{dx}\ln\sqrt{x} =$

Example. $\frac{d}{dx}\ln\left(\frac{x+1}{x-1}\right) =$

Example. $\frac{d}{dx}\ln(\cos x) =$

Example. $\frac{d}{dx}(x^2 - 7x)\log_3 x =$

Example. $\frac{d}{dx}\log_5(\tan x) =$

Example. Find the derivative of $y = \dfrac{e^{3x}\sqrt{x^2+1}}{(x^3+17)^4}$. This would be hard using the quotient rule (which would include a product rule for the derivative of the numerator), but we can simplify work using the trick of "logarithmic differentiation."

Example. Use logarithmic differentiation to find the derivative of $y = x^x$. Same method can be used to find the derivative of any function of form $f(x)^{g(x)}$.

3.5 Inverse Trigonometric Functions

The functions sin, cos and tan are not one-to-one functions, so in order for them to have an inverse, we first make them one-to-one by restricting the domain. The functions arcsin, arccos and arctan are inverses of the functions sin, cos and tan restricted as follows.

We can say: $\arcsin x$ is the angle θ whose sine is x and falls in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arccos x$ is the angle θ whose cosine is x and falls in $[0, \pi]$

$\arctan x$ is the angle θ whose tangent is x and falls in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Derivatives of inverse trigonometric functions.

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

We justify the derivatives of inverse trigonometric functions.

Example. $\dfrac{d}{dx} \arctan(7x) =$

Example. $\dfrac{d}{dx} (\arctan x)^2 =$

Example. $\dfrac{d}{dx} \left(x \arcsin x + \sqrt{1 - x^2}\right) =$

3.7 L'Hospital's Rule

When computing limits, the difficult ones are always an indeterminate form:

$$\infty - \infty \qquad 0 \cdot \infty \qquad \frac{\infty}{\infty} \qquad \frac{0}{0}$$

The rule below helps us find some of them.

Theorem (L'Hospital's Rule). Suppose f and g are differentiable near a and $g'(x) \neq 0$ near a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \text{ if the latter exists, or is } \pm\infty$$

The rule also holds when $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, or the limit is one-sided.

Note. The rule helps with forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Note that this is *not* the quotient rule for derivatives, it is a statement about limits that uses derivatives.

Example. $\lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x}{\sin x - 1} =$

Example. $\lim_{x\to 0} \frac{\tan x - x}{x^3} =$

Example. $\lim_{x\to 0+} x \ln x =$

Exponential indeterminate forms are: 0^0, ∞^0, 1^∞.

Example. $\lim\limits_{x\to 0+} x^{\sqrt{x}} =$

Example. $\lim\limits_{x\to\infty} \left(\dfrac{x}{x+1}\right)^x =$

Example. $\lim\limits_{x\to\infty} \dfrac{x^3}{e^x} =$

Similarly, $\lim\limits_{x\to\infty} \dfrac{x^c}{e^x} = 0$ for any $c > 0$, that is, e^x grows faster than any x^c, $c > 0$, which is interesting for large positive numbers c.

Example. $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} =$

Similarly, $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^c} = 0$ for any $c > 0$, that is, $\ln x$ grows slower than any x^c, $c > 0$, which is interesting for small positive numbers c.