1.3 Limits

Example. Consider the function $f(x) = \frac{\sqrt{x}-2}{x-4}$. This function is clearly not defined at x = 4. What happens when x approaches 4?

Evaluate the function at numbers close to 4 and graph it on an interval around 4.

x	$\frac{\sqrt{x}-2}{x-4}$
4.1	
4.01	
4.001	
3.9	
3.99	
3.999	

It appears that f(x) gets closer and closer to as x gets closer and closer to 4.

We write $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} =$ and say "the limit of $\frac{\sqrt{x}-2}{x-4}$, as x goes to 4, is ."

Example. Consider the function $f(x) = \frac{\sin x}{x}$. What happens when x approaches 0?

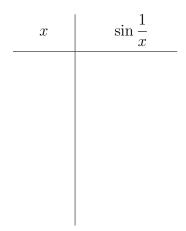
Evaluate the function at numbers close to 0 and graph it on an interval around 0 (radian mode is what we use in calculus!).

x	$\frac{\sin x}{x}$

It appears that f(x) gets closer and closer to as x gets closer and closer to 0, so

$$\lim_{x \to 0} \frac{\sin x}{x} = \qquad .$$

Example. Consider the function $f(x) = \sin \frac{1}{x}$. Where is its behavior interesting? Evaluate the function at appropriate numbers and graph it on an appropriate interval.



Note. $\lim_{x\to a} f(x)$ exists only if values of f(x) approach a single number as x goes to a.

Example. Graph the function

$$f(x) = \begin{cases} x+2 & \text{if } x > 1, \\ -x+1 & \text{if } x < 1 \\ 2 & \text{if } x = 1. \end{cases}$$

What can you say about $\lim_{x\to 1} f(x)$?

Something can be salvaged, though: as x goes to 1 from left, f(x) approaches 0 as x goes to 1 from right, f(x) approaches 3

We write

$$\lim_{x \to 1-} f(x) = 0$$
 and $\lim_{x \to 1+} f(x) = 3$

and call these one-sided limits.

Note. f(1) = 2, but this does not matter when computing $\lim_{x \to 1} f(x)$, $\lim_{x \to 1-} f(x)$ or $\lim_{x \to 1+} f(x)$.

In general, when trying to figure out $\lim_{x\to a} f(x)$, we only consider x's close to a, but not equal to a. f(a) may not even be defined, as in most of our examples.

Calculus 1 — Lecture notes MAT 250, Spring 2024 — D. Ivanšić

1.4 Calculating Limits

Example. (Accuracy.) Investigate $f(x) = (1-x)^{\frac{1}{x}}$ when $x \to 0$.

- a) Sketch the graph of the function around the relevant point.
- b) What is the approximate $\lim_{x\to 0} f(x)$, accurate to six decimal points? Write a table of values that will justify your answer.

Example. (Trust Calculator?) Investigate $f(x) = \frac{5(\sqrt{x^3+4}-2)}{x^3}$ when $x \to 0$.

- a) Sketch the graph of the function. From the graph and numerical evidence, what does $\lim_{x\to 0} f(x)$ appear to be?
- b) Compute the values of f(x) for $x = 10^{-4}, 10^{-5}, \dots, 10^{-8}$. Write the table of values here. What appears to be the limit now?
- c) Try to explain why a) and b) apparently give different answers. (Hint: enter $1+10^{-14}-1$ in your calculator. What is the exact value of this expression? What does the calculator say? What is happening?)

x	$\frac{5(\sqrt{x^3+4}-2)}{x^3}$	x	$\frac{5(\sqrt{x^3+4}-2)}{x^3}$
0.1		-0.1	
0.01		-0.01	
0.001		-0.001	
10^{-4}		-10^{-4}	
10^{-5}		-10^{-5}	
10^{-6}		-10^{-6}	
10^{-7}		-10^{-7}	
10^{-8}		-10^{-8}	

Example. (Limit Laws.) Let $u \to 3$, $v \to 5$. What do u + v, u - v, $u \cdot v$ and $\frac{u}{v}$ approach?

u	v	u + v	u-v	$u \cdot v$	u/v
2.9	4.9				
2.99	4.99				
2.999	4.999				
2.9	5.1				
2.99	5.01				
2.999	5.001				
3.1	4.9				
3.01	4.99				
3.001	4.999				
3.1	5.1				
3.01	5.01				
3.001	5.001				

The table above justifies the following limit laws: if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \quad (1) \qquad \lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \quad (4)$$

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \quad (2) \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \text{ if } \lim_{x \to a} g(x) \neq 0 \quad (5)$$

$$\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x) \quad (3)$$

We also have the following two basic limits that are intuitively clear:

$$\lim_{x \to a} c = c \quad (7) \qquad \qquad \lim_{x \to a} x = a \quad (8)$$

Example. Use limit laws to find the following limits. Mark by number which limit law you are using at every step.

$$\lim_{x \to -1} (x^2 - 3x + 3) =$$

$$\lim_{x \to 2} \frac{x^2 + x}{4x - 1} =$$

The previous two examples show that, due to limit laws, calculating $\lim_{x\to a} f(x)$ amounts to plugging in x=a into the function f(x), when the function is a polynomial or a rational function (in other words, when it is constructed using the operations $+, -, *, \div$).

Direct substitution property. If f(x) is a polynomial or a rational function, and f(a) is defined, then

$$\lim_{x \to a} f(x) = f(a)$$

Note. This property is true also for functions sin, cos, $\sqrt[n]{}$. Two other general rules are

$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n \quad (10) \qquad \qquad \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \quad (11)$$

Examples.

$$\lim_{x \to 3} \sqrt[3]{\frac{3x - 1}{x^2 - x + 4}} =$$

$$\lim_{x \to \pi} \frac{\cos x}{x - \sin x} =$$

Examples. What if evaluation gives us an undefined number?

$$\lim_{x \to -1} \frac{x^2 - 2x - 3}{x + 1} =$$

$$\lim_{x \to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3} =$$

$$\lim_{x \to 0} \frac{5(\sqrt{x^3 + 4} - 2)}{x^3} =$$

$$\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right)=$$

Example. What if limit laws do not apply and algebra is not possible?

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} =$$

Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ on some interval around a (except maybe at a)

and
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then
$$\lim_{x \to a} g(x) = L$$

 $Graphical \ ``proof".$

Use the squeeze theorem to find the limit of the previous example.

Example. Use the squeeze theorem to show $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

Examples. More trigonometric limits.

$$\lim_{x \to 0} \frac{\sin(6x)}{x} =$$

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} =$$

Calculus 1 — Lecture notes
MAT 250, Spring 2024 — D. Ivanšić

1.5 Continuity

A function is continuous at a point a if the graph of f does not have a break at a.

This definition captures the idea:

Definition. A function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

Note. Three things are needed for a function to be continuous at a.

- 1) f is defined at a.
- 2) $\lim_{x\to a} f(x)$ exists (and is a real number).
- $3) \lim_{x \to a} f(x) = f(a)$

(Read about the various types of discontinuities in the book.)

Definition. A function f is continuous on an interval if it is continuous at every point of that interval.

Graphically. A function is continuous on an interval if its graph on that interval can be drawn without lifting pencil from paper.

Theorem. If f and g are continuous at a (or an interval), then the following functions are continuous at a (or an interval):

$$f + g, f - g, f \cdot g, \frac{f}{g} \text{ (if } g(a) \neq 0)$$

Proof for one of the functions.

Theorem. Polynomials, rational functions, root functions, exponential functions and logarithmic functions are continuous where they are defined.

Proof.

Theorem. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b)$$

Example. $\lim_{x\to 3} \sin \frac{x^2 - 5x + 6}{x - 3} =$

Theorem. If g is continuous at a and f is continuous at g(a), then $f \circ g$ is continuous at a.

Example. $e^{\tan x}$ is continuous wherever it is defined since it is a composite of e^x and $\tan x$, functions that are continuous wherever they are defined.

In the same way, using two previous theorems, any *single* formula is continuous wherever it is defined. For example,

$$\sqrt{\frac{\sin x + 4x^{\frac{2}{5}}}{2^x \cdot \ln x}}$$
 is continuous wherever it is defined.

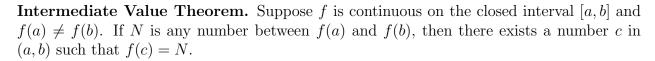
Most physical phenomena are described by continuous functions (unbroken graphs).

Examples. Temperature and position as functions of time.

Examples.

If T(8) = 55°F and T(11) = 75°F, at some time between 8 and 11, temperature was 65°F.

Traveling along a road from point A to point B we must pass through every point E between them.



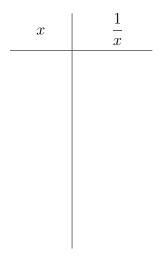
 $Graphical\ "proof".$

Example. Show that the equation $x^3 - 2x^2 + 3x + 1 = 0$ has a solution in the interval [-1, 1]. Then find an interval of width 0.01 that contains the solution.

Calculus 1 — Lecture notes MAT 250, Spring 2024 — D. Ivanšić

1.6 Limits Involving Infinity

Example. Consider the function $f(x) = \frac{1}{x}$ around 0.



We see that f(x) does not approach any *real* number as x approaches 0 from either side, so $\lim_{x\to 0+}\frac{1}{x}$ and $\lim_{x\to 0-}\frac{1}{x}$ do not exist. However, they do not exist in a particular way, namely:

As $x \to 0+$, $\frac{1}{x}$ grows without bound ("goes to ∞ ")

As $x \to 0-$, $\frac{1}{x}$ drops without bound ("goes to $-\infty$ ")

This behavior is written as:

$$\lim_{x\to 0+}\frac{1}{x}=\infty \qquad \qquad \lim_{x\to 0-}\frac{1}{x}=-\infty$$

In general, the table above justifies that

$$\frac{1}{\text{small positive}} = \text{large positive}$$
 $\frac{1}{\text{small negative}} = \text{large negative}$

so if f(x) is any expression,

if
$$f(x) \to 0$$
 and $f(x) > 0$ (written as $f(x) \to 0+$), then $\frac{1}{f(x)} \to \infty$

if
$$f(x) \to 0$$
 and $f(x) < 0$ (written as $f(x) \to 0$ -), then $\frac{1}{f(x)} \to -\infty$

These facts are written in shorthand as $\frac{1}{0+} = \infty$ and $\frac{1}{0-} = -\infty$

Example. Find the limits.

$$\lim_{x\to 2+}\frac{1}{6-3x}=$$

$$\lim_{x\to 2-}\frac{1}{6-3x}=$$

Note. When $\lim_{x\to a} f(x) = \infty$ (or $-\infty$, or same in the case of a one-sided limit), then the line x = a is a vertical asymptote of the graph of f.

Example. Consider the functions of type $f(x) = \frac{1}{x^c}$, (c > 0) and see what happens to values of f(x) as x grows without bound.

x	$\frac{1}{x}$	$\frac{1}{x^2}$	$\frac{1}{\sqrt{x}}$	$\frac{1}{x^c}$

In all cases, values of f(x) approach 0, so we write $\lim_{x\to\infty}\frac{1}{x^c}=0$ for c>0. This is true, essentially, because:

$$\frac{1}{\text{large positive}} = \text{small positive}$$
 $\frac{1}{\text{large negative}} = \text{small negative}$

which gives rise to this shorthand: $\frac{1}{\infty} = 0$ and $\frac{1}{-\infty} = 0$.

Note. When $\lim_{x\to\infty} f(x) = L$ (or $x\to -\infty$), then the line y=L is a horizontal asymptote of the graph of f.

Quintessential Example.

 $f(x) = \arctan x$

Example. Consider the functions of type $f(x) = x^n$, n > 0 integer, and see what happens to values of f(x) as x grows without bound by evaluating and by observing the graphs. More generally, consider functions of type $f(x) = x^c$, c > 0.

x	x^2	x^3	\sqrt{x}	x^c

We see:

$$\lim_{x \to \infty} x^n = \infty \qquad \lim_{x \to -\infty} x^n = \left\{ \begin{array}{cc} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{array} \right. \qquad \lim_{x \to \infty} x^c = \infty \qquad \left(\begin{array}{c} c, n > 0 \\ n \text{ an integer} \end{array} \right)$$

Example.
$$\lim_{x \to \infty} (x^3 - 5x^2 + 3x + 10) =$$

Note. For a general polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $\lim_{x \to \pm \infty} P(x) = \pm \infty$, which depends on the degree and the sign of a_n .

Show this statement for n odd, $a_n < 0, x \to \infty$.

Thus the graphs of polynomials have one of these general shapes:

Example.
$$\lim_{x \to \infty} \frac{5x^2 - 3x + 1}{2x^2 + 4x + 3} =$$

Example.
$$\lim_{x \to \infty} \frac{2x^2 - 7x + 1}{x^3 + 1} =$$

Extended limit laws.
$$\frac{1}{0+} = \infty \qquad \frac{1}{0-} = -\infty \qquad \frac{L}{\pm \infty} = 0$$

$$L \cdot \infty = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases} \qquad \infty + \infty = \infty \qquad L + \infty = \infty$$

$$L - \infty = -\infty$$

Keeping in mind these are shorthand for statements about limits, write out what $L \cdot \infty = \infty$ (L>0) means.

Missing from the list of extended limit laws are the expressions

$$\infty - \infty$$
 $0 \cdot \infty$ $\frac{\infty}{\infty}$ $\frac{0}{0}$

These are called *indeterminate forms*, because the limit cannot be determined just by knowing the limits of f and g.

Example. Show that $0 \cdot \infty$ is indeterminate by providing examples of functions f and g so that in each example $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = \infty$, but $\lim_{x\to 0} f(x)g(x)$ varies. (Think simple.)