1.3 Limits

Example. Consider the function $f(x) = \dfrac{\sqrt{x}-2}{x-4}$. This function is clearly not defined at $x = 4$. What happens when x approaches 4?

Evaluate the function at numbers close to 4 and graph it on an interval around 4.

x	$\dfrac{\sqrt{x}-2}{x-4}$
4.1	
4.01	
4.001	
3.9	
3.99	
3.999	

It appears that $f(x)$ gets closer and closer to   as x gets closer and closer to 4.

We write $\displaystyle\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} =$ and say "the limit of $\dfrac{\sqrt{x}-2}{x-4}$, as x goes to 4, is   ."

Example. Consider the function $f(x) = \dfrac{\sin x}{x}$. What happens when x approaches 0?

Evaluate the function at numbers close to 0 and graph it on an interval around 0 (radian mode is what we use in calculus!).

x	$\dfrac{\sin x}{x}$

It appears that $f(x)$ gets closer and closer to   as x gets closer and closer to 0, so

$\displaystyle\lim_{x\to 0} \frac{\sin x}{x} =$.

Example. Consider the function $f(x) = \sin\frac{1}{x}$. Where is its behavior interesting? Evaluate the function at appropriate numbers and graph it on an appropriate interval.

x	$\sin\frac{1}{x}$

Note. $\lim_{x\to a} f(x)$ exists only if values of $f(x)$ approach *a single number* as x goes to a.

Example. Graph the function

$$f(x) = \begin{cases} x+2 & \text{if } x > 1, \\ -x+1 & \text{if } x < 1 \\ 2 & \text{if } x = 1. \end{cases}$$

What can you say about $\lim_{x\to 1} f(x)$?

Something can be salvaged, though: as x goes to 1 from left, $f(x)$ approaches 0
as x goes to 1 from right, $f(x)$ approaches 3

We write

$$\lim_{x\to 1-} f(x) = 0 \text{ and } \lim_{x\to 1+} f(x) = 3$$

and call these *one-sided limits.*

Note. $f(1) = 2$, but this does not matter when computing $\lim_{x\to 1} f(x)$, $\lim_{x\to 1-} f(x)$ or $\lim_{x\to 1+} f(x)$.

In general, when trying to figure out $\lim_{x\to a} f(x)$, *we only consider x's close to a, but not equal to a.* $f(a)$ may not even be defined, as in most of our examples.

1.4 Calculating Limits

Example. *(Accuracy.)* Investigate $f(x) = (1-x)^{\frac{1}{x}}$ when $x \to 0$.
a) Sketch the graph of the function around the relevant point.
b) What is the approximate $\lim_{x\to 0} f(x)$, *accurate to six decimal points*? Write a table of values that will justify your answer.

Example. *(Trust Calculator?)* Investigate $f(x) = \dfrac{5(\sqrt{x^3+4}-2)}{x^3}$ when $x \to 0$.
a) Sketch the graph of the function. From the graph and numerical evidence, what does $\lim_{x\to 0} f(x)$ appear to be?
b) Compute the values of $f(x)$ for $x = 10^{-4}, 10^{-5}, \ldots, 10^{-8}$. Write the table of values here. What appears to be the limit now?
c) Try to explain why a) and b) apparently give different answers. (Hint: enter $1 + 10^{-14} - 1$ in your calculator. What is the exact value of this expression? What does the calculator say? What is happening?)

x	$\dfrac{5(\sqrt{x^3+4}-2)}{x^3}$	x	$\dfrac{5(\sqrt{x^3+4}-2)}{x^3}$
0.1		-0.1	
0.01		-0.01	
0.001		-0.001	
10^{-4}		-10^{-4}	
10^{-5}		-10^{-5}	
10^{-6}		-10^{-6}	
10^{-7}		-10^{-7}	
10^{-8}		-10^{-8}	

Example. *(Limit Laws.)* Let $u \to 3$, $v \to 5$. What do $u+v$, $u-v$, $u \cdot v$ and $\frac{u}{v}$ approach?

u	v	$u+v$	$u-v$	$u \cdot v$	u/v
2.9 2.99 2.999	4.9 4.99 4.999				
2.9 2.99 2.999	5.1 5.01 5.001				
3.1 3.01 3.001	4.9 4.99 4.999				
3.1 3.01 3.001	5.1 5.01 5.001				

The table above justifies the following limit laws: if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \quad (1)$$

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \quad (4)$$

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \quad (2)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0 \quad (5)$$

$$\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x) \quad (3)$$

We also have the following two basic limits that are intuitively clear:

$$\lim_{x\to a} c = c \quad (7) \qquad\qquad \lim_{x\to a} x = a \quad (8)$$

Example. Use limit laws to find the following limits. Mark by number which limit law you are using at every step.

$$\lim_{x\to -1}(x^2-3x+3) =$$

$$\lim_{x\to 2}\frac{x^2+x}{4x-1} =$$

The previous two examples show that, due to limit laws, calculating $\lim_{x\to a} f(x)$ amounts to plugging in $x = a$ into the function $f(x)$, when the function is a polynomial or a rational function (in other words, when it is constructed using the operations $+, -, *, \div$).

Direct substitution property. If $f(x)$ is a polynomial or a rational function, and $f(a)$ is defined, then

$$\lim_{x\to a} f(x) = f(a)$$

Note. This property is true also for functions sin, cos, $\sqrt[n]{}$. Two other general rules are

$$\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n \quad (10) \qquad \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} \quad (11)$$

Examples.

$$\lim_{x\to 3} \sqrt[3]{\frac{3x-1}{x^2-x+4}} =$$

$$\lim_{x\to \pi} \frac{\cos x}{x - \sin x} =$$

Examples. What if evaluation gives us an undefined number?

$$\lim_{x\to -1} \frac{x^2-2x-3}{x+1} =$$

$$\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} =$$

$$\lim_{x\to 0} \frac{5(\sqrt{x^3+4}-2)}{x^3} =$$

$$\lim_{x \to 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x - 2} \right) =$$

Example. What if limit laws do not apply and algebra is not possible?

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} =$$

Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ on some interval around a (except maybe at a)

$$\text{and } \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

$$\text{then } \lim_{x \to a} g(x) = L$$

Graphical "proof".

Use the squeeze theorem to find the limit of the previous example.

Example. Use the squeeze theorem to show $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

Examples. More trigonometric limits.

$$\lim_{x \to 0} \frac{\sin(6x)}{x} =$$

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} =$$

1.5 Continuity

A function is continuous at a point a if the graph of f does not have a break at a.

This definition captures the idea:

Definition. A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

Note. Three things are needed for a function to be continuous at a.

1) f is defined at a.

2) $\lim_{x \to a} f(x)$ exists (and is a real number).

3) $\lim_{x \to a} f(x) = f(a)$

(Read about the various types of discontinuities in the book.)

Definition. A function f is continuous on an interval if it is continuous at every point of that interval.

Graphically. A function is continuous on an interval if its graph on that interval can be drawn without lifting pencil from paper.

Theorem. If f and g are continuous at a (or an interval), then the following functions are continuous at a (or an interval):

$$f+g, f-g, f\cdot g, \frac{f}{g} \text{ (if } g(a)\neq 0)$$

Proof for one of the functions.

Theorem. Polynomials, rational functions, root functions, exponential functions and logarithmic functions are continuous where they are defined.

Proof.

Theorem. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)$$

Example. $\lim_{x\to 3} \sin \dfrac{x^2-5x+6}{x-3} =$

Theorem. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Example. $e^{\tan x}$ is continuous wherever it is defined since it is a composite of e^x and $\tan x$, functions that are continuous wherever they are defined.

In the same way, using two previous theorems, any *single* formula is continuous wherever it is defined. For example,

$$\sqrt{\frac{\sin x + 4x^{\frac{2}{5}}}{2^x \cdot \ln x}} \text{ is continuous wherever it is defined.}$$

Most physical phenomena are described by continuous functions (unbroken graphs).

Examples. Temperature and position as functions of time.

Examples.

If $T(8) = 55°\text{F}$ and $T(11) = 75°\text{F}$, at some time between 8 and 11, temperature was 65°F.

Traveling along a road from point A to point B we must pass through every point E between them.

Intermediate Value Theorem. Suppose f is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$. If N is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) such that $f(c) = N$.

Graphical "proof".

Example. Show that the equation $x^3 - 2x^2 + 3x + 1 = 0$ has a solution in the interval $[-1, 1]$. Then find an interval of width 0.01 that contains the solution.

1.6 Limits Involving Infinity

Example. Consider the function $f(x) = \dfrac{1}{x}$ around 0.

x	$\dfrac{1}{x}$

We see that $f(x)$ does not approach any *real* number as x approaches 0 from either side, so $\lim\limits_{x\to 0+} \dfrac{1}{x}$ and $\lim\limits_{x\to 0-} \dfrac{1}{x}$ do not exist. However, they do not exist in a particular way, namely:

As $x \to 0+$, $\dfrac{1}{x}$ grows without bound ("goes to ∞")

As $x \to 0-$, $\dfrac{1}{x}$ drops without bound ("goes to $-\infty$")

This behavior is written as:

$$\lim_{x\to 0+} \frac{1}{x} = \infty \qquad\qquad \lim_{x\to 0-} \frac{1}{x} = -\infty$$

In general, the table above justifies that

$$\frac{1}{\text{small positive}} = \text{large positive} \qquad\qquad \frac{1}{\text{small negative}} = \text{large negative}$$

so if $f(x)$ is any expression,

$$\text{if } f(x) \to 0 \text{ and } f(x) > 0 \quad (\text{written as } f(x) \to 0+), \text{ then } \frac{1}{f(x)} \to \infty$$

$$\text{if } f(x) \to 0 \text{ and } f(x) < 0 \quad (\text{written as } f(x) \to 0-), \text{ then } \frac{1}{f(x)} \to -\infty$$

These facts are written in shorthand as $\dfrac{1}{0+} = \infty$ and $\dfrac{1}{0-} = -\infty$

Example. Find the limits.

$$\lim_{x\to 2+} \frac{1}{6-3x} =$$

$$\lim_{x\to 2-} \frac{1}{6-3x} =$$

Note. When $\lim_{x\to a} f(x) = \infty$ (or $-\infty$, or same in the case of a one-sided limit), then the line $x = a$ is a vertical asymptote of the graph of f.

Example. Consider the functions of type $f(x) = \dfrac{1}{x^c}$, $(c > 0)$ and see what happens to values of $f(x)$ as x grows without bound.

x	$\frac{1}{x}$	$\frac{1}{x^2}$	$\frac{1}{\sqrt{x}}$	$\frac{1}{x^c}$

In all cases, values of $f(x)$ approach 0, so we write $\lim_{x\to\infty} \frac{1}{x^c} = 0$ for $c > 0$. This is true, essentially, because:

$$\frac{1}{\text{large positive}} = \text{small positive} \qquad \frac{1}{\text{large negative}} = \text{small negative}$$

which gives rise to this shorthand: $\dfrac{1}{\infty} = 0$ and $\dfrac{1}{-\infty} = 0$.

Note. When $\lim_{x\to\infty} f(x) = L$ (or $x \to -\infty$), then the line $y = L$ is a horizontal asymptote of the graph of f.

Quintessential Example.

$f(x) = \arctan x$

Example. Consider the functions of type $f(x) = x^n$, $n > 0$ integer, and see what happens to values of $f(x)$ as x grows without bound by evaluating and by observing the graphs. More generally, consider functions of type $f(x) = x^c, c > 0$.

x	x^2	x^3	$\sqrt{x}$	x^c

We see:

$$\lim_{x\to\infty} x^n = \infty \qquad \lim_{x\to-\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases} \qquad \lim_{x\to\infty} x^c = \infty \quad \begin{pmatrix} c, n > 0 \\ n \text{ an integer} \end{pmatrix}$$

Example. $\lim_{x\to\infty}(x^3 - 5x^2 + 3x + 10) =$

Note. For a general polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $\lim_{x\to\pm\infty} P(x) = \pm\infty$, which depends on the degree and the sign of a_n.

Show this statement for n odd, $a_n < 0$, $x \to \infty$.

Thus the graphs of polynomials have one of these general shapes:

Example. $\lim_{x\to\infty} \dfrac{5x^2 - 3x + 1}{2x^2 + 4x + 3} =$

Example. $\lim\limits_{x\to\infty} \dfrac{2x^2 - 7x + 1}{x^3 + 1} =$

Extended limit laws. $\dfrac{1}{0+} = \infty$ $\quad \dfrac{1}{0-} = -\infty$ $\quad \dfrac{L}{\pm\infty} = 0$

$$L \cdot \infty = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases} \qquad \begin{array}{l} \infty + \infty = \infty \\ \infty \cdot \infty = \infty \end{array} \qquad \begin{array}{l} L + \infty = \infty \\ L - \infty = -\infty \end{array}$$

Keeping in mind these are shorthand for statements about limits, write out what $L \cdot \infty = \infty$ $(L > 0)$ means.

Missing from the list of extended limit laws are the expressions

$$\infty - \infty \qquad 0 \cdot \infty \qquad \frac{\infty}{\infty} \qquad \frac{0}{0}$$

These are called *indeterminate forms*, because the limit cannot be determined just by knowing the limits of f and g.

Example. Show that $0 \cdot \infty$ is indeterminate by providing examples of functions f and g so that in each example $\lim\limits_{x\to 0} f(x) = 0$, $\lim\limits_{x\to 0} g(x) = \infty$, but $\lim\limits_{x\to 0} f(x)g(x)$ varies. (Think simple.)