Integration Theory — Handout
MAT 725, Fall 2024 — D. Ivanšić

Test Knowledge

Sections 8.1-8.4

Definitions

Pointwise convergence of a sequence of functions (8.1.1)
Uniform convergence of a sequence of functions (8.1.4)
Uniform norm of a bounded function (8.1.7)
Definition of the exponential function (8.3.1)
Definition of the logarithmic function (8.3.8)
Definition of e (8.3.5)
Definition of the general power function (8.3.10)
Definition of cos and sin (8.4.1)

Theorems

Lemma 8.1.5
Lemma 8.1.8
Cauchy Criterion for Uniform Convergence (8.1.10)
Interchange of Limit and Continuity (Theorem 8.2.2)
Interchange of Limit and Derivative (Theorem 8.2.3)
Interchange of Limit and Integral (Theorem 8.2.4)
Properties of the exponential function ((i)–(vi) in 8.3.1, 8.3.6 and 8.3.7)
Corollary 8.3.3
Uniqueness of the exponential function (8.3.4)
Properties of the logarithmic function (8.3.9)
Properties of the general power function (8.3.11, 8.3.12)
Properties of cos and sin ((i)–(vi) in 8.4.1, 8.4.2, 8.4.3)
Uniqueness of cos and sin (8.4.4)

Proofs

Cauchy Criterion for Uniform Convergence (8.1.10)
Interchange of Limit and Continuity (8.2.2)
Interchange of Limit and Integral (8.2.4)
Existence of the exponential function (8.3.1)
Corollary 8.3.3
Uniqueness of the exponential function (8.3.4)

Integration Theory — Handout
MAT 725, Fall 2024 — D. Ivanšić

Test Knowledge

Sections 1.1–1.4

Definitions

Definition of a function of bounded variation (section 1.1)
Various examples of functions of bounded and unbounded variation
Definition of a curve in a plane or space
Definition of a rectifiable curve
Riemann-Stieltjes Integral (1.3)
Open set, point of closure, closure, closed set (1.4)
Cover, open cover, finite cover (1.4)
Compact set (1.4)
Interior of a set (1.4)
σ-algebra, smallest σ-algebra containing a collection (1.4)
G_δ, F_σ, Borel sets (1.4)

Theorems

Monotone and Lipschitz functions are of bounded variation (section 1.1)
Theorems 1.1, 1.2
Jordan's Theorem (1.7)
Theorems 1.8, 1.9
Corollary 1.10 (only for V)
Theorem 1.13
Theorem: $L(C) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$ (section 1.2)
Cauchy's criterion for R-S integral existence (1.3)
Common discontinuity of f and ϕ causes no R-S integral (1.3)
Theorems 1.16 and 1.17 in 1.3
Integration by parts (Theorem 1.21)
Sufficient condition for existence of R-S integral and bound (Theorem 1.24)
Mean Value Theorem for R-S integral (1.27)
If f and ϕ' are both continuous, $\int_a^b f\, d\phi = \int_a^b f\phi'$ (1.3)
Open set is a union of countably many disjoint open intervals (Prop. 1.9)
Statements on unions and intersections of open or closed sets (Prop. 1.8, 1.12)
Propositions 1.10 and 1.11 in 1.4
Heine-Borel Theorem (1.4)
Nested Set Theorem (1.4)
Proposition 1.13 in 1.4

Proofs

Theorem 1.2
If f and ϕ' are both continuous, $\int_a^b f\, d\phi = \int_a^b f\phi'$ (1.3)
Propositions 1.10 and 1.11 in 1.4

Integration Theory — Handout
MAT 725, Fall 2024 — D. Ivanšić

Test Knowledge

Sections 2.1–2.7

Definitions

Lebesgue measure (2.1)
Lebesgue outer measure (2.2)
Outer measure of a set (2.2)
Measurable set (2.3)
Lebesgue measure as the restriction of outer measure on measurable sets (2.5)
Cantor set (2.7)
Cantor-Lebesgue function (2.7)

Theorems

Outer measure equals length for intervals (Proposition 2.1)
Outer measure is translation-invariant and subadditive (Propositions 2.2, 2.3)
A countable set has outer measure 0 (2.2)
Any set of outer measure 0 is measurable (Prop 2.4)
Finite unions of measurable sets are measurable (Prop. 2.5)
Outer measure is finitely additive on measurable sets (Prop 2.6)
Countable unions of measurable sets are measurable (Prop. 2.7)
Every interval is measurable (Prop. 2.8)
Collection of measurable sets is a σ-algebra (Prop. 2.9)
Translate of a measurable set is measurable (Prop. 2.10)
Outer and inner approximation of measurable sets (Theorem 2.11)
A measurable set is a G_δ-set with a set of outer measure 0 removed (2.4)
A measurable set is an F_σ-set with a set of outer measure 0 added (2.4)
Lebesgue measure is countably additive (Prop. 2.13)
A Lebesgue measure derived from outer measure
is a Lebesgue measure in the sense of 2.1 (Theorem 2.14)
Continuity of measure over ascending and descending collection (Theorem 2.15)
The Borel-Cantelli Lemma (2.5)
Lemma 2.16
Every set of measure > 0 has a nonmeasureable subset (Vitali's Theorem 2.17)
Theorem 2.18
Cantor set is closed, countable and has measure 0 (Prop. 2.19)
Cantor-Lebesgue function is increasing and continuous (Prop. 2.20)
Continuous bijection ψ maps a set of measure 0 to a set of nonzero measure,
maps a measurable set to a nonmeasurable set (Prop. 2.21)
There exists a measurable set that is not Borel (Prop. 2.22)

Proofs

A countable set has outer measure 0 (2.2)
Subadditivity of outer measure (Proposition 2.3)
Finite unions of measurable sets are measurable (Prop. 2.5)
Countable unions of measurable sets are measurable (Prop. 2.7)
Continuity of measure over ascending and descending collection (Theorem 2.15)
Lemma 2.16
Every set of measure > 0 has a nonmeasureable subset (Vitali's Theorem 2.17)
Cantor set is closed, countable and has measure zero (Prop. 2.19)