Integration Theory — Exam 1
MAT 725, Fall 2024 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

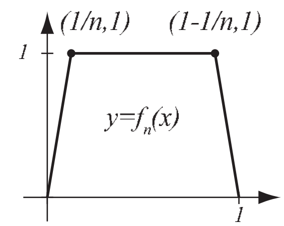
Theory 1. (3pts) Define uniform convergence for a sequence of functions.

Theory 2. (3pts) State the theorem on uniform convergence and continuity.

Theory 3. (3pts) State Jordan's theorem about functions of bounded variation.

Type A problems (5pts each)

A1. Let $f_n : [0,1] \to \mathbf{R}$ be the function whose graph is at right.
a) Find the function f that is the limit of the sequence (f_n).
b) Draw a picture to show that (f_n) does not uniformly converge to f on $[0,1]$.



A2. For the sequence of functions $f_n(x) = \cos(\frac{1}{n}x)$ on the interval $[0, 2\pi]$, do the following:
a) Verify that it satisfies assumptions of the theorem on interchange of limit and derivative.
b) Verify that it satisfies the conclusion of the theorem on interchange of limit and derivative.

A3. As previously defined, let $x^a = E(aL(x))$, where E, L are the exponential and logarithmic functions. Use properties of E and L to show 1) $x^a x^b = x^{a+b}$ 2) $(x^a)^b = x^{ab}$.

A4. Find the variation of the function $f(x) = x^2 - 4x$ on the interval $[1, 5]$.

A5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a function satisfying $f''(x) = f(x)$. Show that $f(x)^2 - f'(x)^2$ is a constant.

Type B problems (8pts each)

B1. Let $f_n : \mathbf{R} \to \mathbf{R}$ be the sequence of functions given by $f_n(x) = xe^{-nx}$. Show that
a) (f_n) converges pointwise to a function f.
b) (f_n) converges uniformly on $[0, \infty)$ by examining $||f_n - f||$. To get $||f_n - f||$, use a calculus 1 technique to find the maximum.

B2. Find a rational number (it doesn't have to be simplified to form $\frac{m}{n}$) that approximates $\sqrt{e}$ with accuracy 10^{-4}.

B3. Suppose $f : \mathbf{R} \to \mathbf{R}$ has the property $f''(x) = -k^2 f(x)$. Show that

$$f(x) = \alpha C(kx) + \beta S(kx) \text{ for some } \alpha, \beta \in \mathbf{R}$$

(Hint: let $g(x) = f(\frac{1}{k}x)$. Show that $g''(x) = -g(x)$, so it follows that $g(x) = \alpha C(x) + \beta S(x)$ for some $\alpha, \beta \in \mathbf{R}$.)

B4. Show that $\displaystyle\lim_{n\to\infty} \int_1^3 \arctan(nx)\, dx = \pi$.

B5. Suppose $f : [c,d] \to \mathbf{R}$ is of bounded variation, and $g : [a,b] \to [c,d]$ is an increasing function (but it doesn't necessarily mean that $g(a) = c$ or $g(b) = d$). Show that $f \circ g : [a,b] \to \mathbf{R}$ is of bounded variation.

B6. Give a simple example showing that if in **B5** we drop the assumption that g is increasing (or monotone) the conclusion does not hold.

Type C problems (12pts each)

C1. Determine if the function $f(x) = x \sin \frac{1}{x}$ for $x \in (0, 1]$, $f(0) = 0$, is of bounded variation on $[0, 1]$.

Integration Theory — Exam 2 **MAT 725, Fall 2024 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) If $\mathbf{r}(t)$, $t \in [a,b]$, is a parametrization of a curve in $\mathbf{R}^3$, define the length of the curve $L(C)$.

Theory 2. (3pts) Define a compact set.

Theory 3. (3pts) State the theorem on semiadditivity of outer measure.

Type A problems (5pts each)

A1. Calculate $\int_2^5 e^{2x}\, dx^2$.

A2. Curve C is given by $\mathbf{r} : [0,4] \to \mathbf{R}^3$, $\mathbf{r}(t) = \left(\frac{t^2}{2}, \frac{2\sqrt{6}}{3} t^{\frac{3}{2}}, 3t\right)$. Calculate the length of C.

A3. If f is constant, determine $\int_a^b f\, d\varphi$.

A4. Let $A = \bigcup_{k=0}^{\infty} [2k, 2k+1)$. Determine $\operatorname{Int} A$ and $\overline{A}$ with explanation.

A5. Give an example of two sets A, B so that $\operatorname{Int}(A \cup B) \neq \operatorname{Int} A \cup \operatorname{Int} B$.

A6. Show that every countable set has outer measure zero.

A7. Let $m : \mathcal{A} \to [0, \infty]$ be a measure, $A, B \in \mathcal{A}$. If $m(A \cap B)$ is finite, show that $m(B - A) = mB - m(A \cap B)$.

Type B problems (8pts each)

B1. Suppose $\int_{-a}^{a} f\, d\varphi$ exists. If f and φ are even functions (g is even if $g(-x) = g(x)$), show that $\int_{-a}^{a} f\, d\varphi = 0$.

B2. Show that for any two sets $A, B \subseteq \mathbf{R}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

B3. Let A be any subset of $\mathbf{R}$, $k > 0$. If $kA = \{kx \mid x \in A\}$, show that $m^*(kA) = k \cdot m^* A$ (use the definition).

B4. Show that the union of finitely many compact sets is a compact set. Then give an example to show the union of countably many compact sets need not be compact.

B5. Let $\mathcal{F} = \{[a,c) \cup (c,b] \mid a, b, c \in \mathbf{R}, a < c < b\}$. Show that the smallest σ-algebra that contains $\mathcal{F}$ is the Borel sets. (Useful fact: every closed interval $[a,b]$ is a union of two elements of $\mathcal{F}$ — how?)

B6. Let $m : \mathcal{A} \to [0, \infty]$ be a measure, $\{E_k, k \in \mathbf{N}\}$ a collection of sets in the σ-algebra $\mathcal{A}$. Show that $m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} mE_k$.

Integration Theory — Exam 3 **MAT 725, Fall 2024 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Define a measurable set.

Theory 2. (3pts) State the theorem on inner approximation by F_σ sets.

Theory 3. (3pts) State the theorem on finite additivity for measurable sets.

Type A problems (5pts each)

A1. Let $E = [0,1] - \left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$. Show that E is measurable and determine m^*E.

A2. Let A have positive outer measure. Show there is an interval $[c,d]$ of length 1 such that $A \cap [c,d]$ has positive outer measure.

A3. Let E be measurable with $m^*E < \infty$ and let $A \supseteq E$ be any set. Prove the excision property: $m^*(A - E) = m^*A - m^*E$.

A4. Show that E is measurable if and only if there exists a G_δ-set G and an F_σ-set F such that $F \subseteq E \subseteq G$ and $m^*(G - F) = 0$.

A5. Let E have finite outer measure. Show that E is measurable if and only if there exists an F_σ-set $F \subset E$ such that $m^*F = m^*E$.

A6. Show that if we remove a measurable subset from a nonmeasurable set, the resulting set is nonmeasurable.

Type B problems (8pts each)

B1. Give an example of an open unbounded set that has finite nonzero measure.

B2. Suppose E has property $\mathcal{I}$: for every $\varepsilon > 0$ there exists a closed set $F \subseteq E$ such that $m^*(E - F) < \varepsilon$. Show directly that if E_1 and E_2 have property $\mathcal{I}$, then $E_1 \cup E_2$ also has property $\mathcal{I}$.

B3. Let E have finite outer measure. Show that there exists a G_δ-set $G \supseteq E$ such that $m^*G = m^*E$.

B4. Define $m^{**}A = \inf\{m^*U \mid A \subseteq U,\ U \text{ open}\}$. Show that $m^*A = m^{**}A$ for every $A \subseteq \mathbf{R}$ by showing $m^*A \leq m^{**}A$ and $m^*A \geq m^{**}A$. The first inequality is obvious (why?) and the second one follows from the definition of m^*A.

B5. We know that if a set E is measurable, then $E = F \cup Z$, where F is an F_σ-set and Z is a set of measure zero. Use this fact to directly prove that the intersection of measurable sets is measurable. Note that you will have to show that the intersection of two F_σ-sets is an F_σ-set.

B6. Show: if E is measurable and has finite measure, then for every $\epsilon > 0$, E is the union of finitely many measurable sets of measure $< \epsilon$.