Integration Theory — Exam 3 **MAT 725, Fall 2015 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Define a measurable set.

Theory 2. (3pts) State the theorem on outer approximation by open sets.

Theory 3. (3pts) State Vitali's theorem on existence of nonmeasurable sets.

Type A problems (5pts each)

A1. Let A be any subset of $\mathbf{R}$. If $-A = \{-x \mid x \in A\}$, show that $m^*(-A) = m^*A$

A2. Let E be measurable and $E \subseteq [0,1]$. Set $E_k = E \cap \left(\frac{1}{k}, 1 - \frac{1}{k}\right)$. What is $\lim\limits_{k\to\infty} mE_k$?

A3. Let $\{E_k \mid k \in \mathbf{N}\}$ be a collection of measurable sets. Show that $\bigcup\limits_{k=1}^{\infty} E_k$ can be written as a disjoint union of measurable sets.

A4. Let E have finite outer measure. Show that E is measurable if and only if there exists an F_σ-set $F \subset E$ such that $m^*F = m^*E$.

A5. Show that a nonmeasurable set must be uncountable.

A6. Show that every nonmeasurable set has a proper subset that is nonmeasurable.

Type B problems (8pts each)

B1. Show that the smallest σ-algebra containing intervals of form $(a, b]$, $a, b < \infty$, is the Borel sets.

B2. Give an example of an open unbounded set that has finite measure.

B3. Suppose E is measurable and has finite measure. Show that for every $\epsilon > 0$ there exists an $n \in \mathbf{N}$ such that $m(E - [-n, n]) < \epsilon$. (In other words, most of E's measure is in a bounded interval.)

B4. We have shown in that for any set E with finite outer measure there is a G_δ-set $G \supseteq E$ such that $m^*G = m^*E$. Use this fact to show that when E is not measurable, it fails the definition using a set A that may be taken to be a G_δ-set.

B5. Show that a union of two measurable sets is measurable either by definition or by using inner approximation by closed sets.

B6. Show: if E is measurable and has finite measure, then for every $\epsilon > 0$, E is union of finitely many measurable sets of measure $< \epsilon$.

Type C problems (12pts each)

C1. Let E be a measurable set with finite measure M. Define $f : \mathbf{R} \to [0, M]$ by setting $f(x) = m((-\infty, x] \cap E)$.
a) Show that f is increasing and the codomain is correct.
b) Show that f is continuous — don't do anything difficult, because it's not.
c) Show that the range of f contains the open interval $(0, M)$. In particular, there exists a set whose measure is $M/2$.
d) Give an example of a set E where the range equals $(0, M)$, so $f(x) \neq 0, M$ for all $x \in \mathbf{R}$.