| **Integration Theory — Exam 1**<br>**MAT 725, Fall 2015 — D. Ivanšić** | **Name:** ______________________<br>*Show all your work!* |
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*Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define the uniform norm of a bounded function.

**Theory 2.** (3pts) State the theorem on interchange of limit and integral.

**Theory 3.** (3pts) Define when a function $f : [a, b] \to \mathbf{R}$ is of bounded variation.

## Type A problems (5pts each)

**A1.** Show that the sequence of functions $f_n(x) = \dfrac{2^{-nx}}{1 + nx^2}$ does not converge uniformly to 0 on $[0, 1]$.

**A2.** Show that $\displaystyle\lim_{n\to\infty} \int_0^1 \sin\left(\frac{x^2}{n}\right) dx = 0$.

**A3.** As previously defined, let $x^a = E(aL(x))$, where $E$, $L$ are the exponential and logarithmic functions. Show 1) $(xy)^a = x^a y^a$ 2) $L(x^a) = aL(x)$.

**A4.** Find the variation of the function $f(x) = \sin x$ on the interval $[0, 4\pi]$.

**A5.** The curve $C$ is given by $\mathbf{r} : [0, 1] \to \mathbf{R}^3$, $\mathbf{r}(t) = (5t, 2t^{\frac{3}{2}}, t^2)$. Show that $C$ is rectifiable and give an upper bound for its length.

## Type B problems (8pts each)

**B1.** Let $f_n : \mathbf{R} \to \mathbf{R}$ be the sequence of functions given by $f_n(x) = \dfrac{1}{1 + n^2x^2}$. Show that
a) $(f_n)$ converges pointwise to a function $f$.
b) $(f_n)$ converges uniformly on $[a, \infty)$ for every $a > 0$.
c) $(f_n)$ does not converge uniformly on $[0, \infty)$.

**B2.** Find a rational number (it doesn't have to be simplified to form $\frac{m}{n}$) that approximates $\sqrt[3]{e}$ with accuracy $10^{-4}$.

**B3.** Suppose $f : \mathbf{R} \to \mathbf{R}$ has the properties: $f''(x) = \frac{1}{2} f(x)$, and $f(0) = f'(0) = 0$. Show that $f(x) = 0$ for all $x \in \mathbf{R}$.

**B4.** Show: if $f : [a, b] \to \mathbf{R}$ is of bounded variation, then so is $f^2$, where $f^2(x) = (f(x))^2$.

**B5.** Suppose $g : \mathbf{R} \to \mathbf{R}$ has a continuous derivative, and $f : [a, b] \to \mathbf{R}$ is of bounded variation. Show that $g \circ f : [a, b] \to \mathbf{R}$ is of bounded variation.

## Type C problems (12pts each)

**C1.** Does the converse to **B4** hold: if $f^2$ is of bounded variation, is $f$?